A GENERALIZATION OF THE $\mathbb{G}_0$ DICHOTOMY AND A STRENGTHENING OF THE $\mathbb{E}_0^{\mathbb{N}}$ DICHOTOMY

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Abstract. We generalize the $\mathbb{G}_0$ dichotomy to doubly-indexed sequences of analytic digraphs. Under a mild definability assumption, we use this generalization to characterize the family of Borel actions of tsi Polish groups on Polish spaces that can be decomposed into countably-many Borel actions admitting complete Borel sets that are lacunary with respect to an open neighborhood of the identity. We also show that if the group in question is non-archimedean, then the inexistence of such a decomposition yields a special kind of continuous embedding of $\mathbb{E}_0^{\mathbb{N}}$ into the corresponding orbit equivalence relation.

Introduction

A *digraph* on a set X is an irreflexive set $G \subseteq X \times X$. The *restriction* of such a digraph to a set $Y \subseteq X$ is given by $G \restriction Y = G \cap (Y \times Y)$. A set $Y \subseteq X$ is *G-independent* if $G \restriction Y = \emptyset$. A *$Z$-coloring* of G is a function $\pi \colon X \to Z$ such that $\pi^{-1}(\{z\})$ is G-independent for all $z \in Z$.

A *homomorphism* from a binary relation R on X to a binary relation S on Y is a function $\phi \colon X \to Y$ such that $w \mathrel{R} x \implies \phi(w) \mathrel{S} \phi(x)$ for all $w, x \in X$. A *homomorphism* from a sequence $(R_i)_{i \in I}$ of binary relations on X to a sequence $(S_i)_{i \in I}$ of binary relations on Y is a function $\phi \colon X \to Y$ that is a homomorphism from R_i to S_i for all $i \in I$.

For all sets N, let X^N denote the set of functions $s \colon N \to X$, and define $X^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N}} X^n$ and $X^{\leq \mathbb{N}} = X^{<\mathbb{N}} \cup X^{\mathbb{N}}$. Given $M \subseteq N$, $s \in X^M$, and $t \in X^N$, we write $s \sqsubseteq t$ to indicate that $s = t \restriction M$. For all $x \in X$, let (x) denote the element of X^1 sending 0 to x. Let $s \frown t$ denote the concatenation of sequences $s \in X^{<\mathbb{N}}$ and $t \in X^{\leq \mathbb{N}}$.

Fix $k_n \in \mathbb{N}$ such that $k_0 = 0$, $\forall n \in \mathbb{N}\ k_{n+1} \leq \max\{k_m \mid m \leq n\} + 1$, and $\forall k \in \mathbb{N} \exists^\infty n \in \mathbb{N}\ k = k_n$, as well as sequences $s_n \in 2^n$ such that $\forall k \in \mathbb{N} \forall s \in 2^{<\mathbb{N}} \exists n \in \mathbb{N}\ (k = k_n \text{ and } s \sqsubseteq s_n)$. For all $s \in 2^{<\mathbb{N}}$, let $\mathbb{G}_s$ denote the digraph on $2^{\mathbb{N}}$ given by $\mathbb{G}_s = \{(s \frown (i) \frown c)_{i<2} \mid c \in 2^{\mathbb{N}}\}$.

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For all $k \in \mathbb{N}$, let $\mathbb{G}_{0,k}$ denote the digraph on $2^{\mathbb{N}}$ given by $\mathbb{G}_{0,k} = \bigcup\{\mathbb{G}_{s_n} \mid n \in \mathbb{N} \text{ and } k = k_n\}$.

Endow $\mathbb{N}$ with the discrete topology, and $\mathbb{N}^{\mathbb{N}}$ with the corresponding product topology. A topological space is *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$, and *Polish* if it is separable and admits a compatible complete metric. A subset of a topological space is *Borel* if it is in the smallest σ-algebra containing the open sets, and *co-analytic* if its complement is analytic. Every Polish space is analytic (see, for example, [Kec95, Theorem 7.9]), and Souslin's theorem ensures that a subset of an analytic Hausdorff space is Borel if and only if it is analytic and co-analytic (see, for example, the proof of [Kec95, 14.11]). A function between topological spaces is *Borel* if preimages of open sets are Borel.

A sequence $(X_{i,j})_{i,j \in \mathbb{N}}$ of sets is *increasing in j* if $X_{i,j} \subseteq X_{i,j+1}$ for all $i, j \in \mathbb{N}$. A digraph G on a topological space X has *countable Borel chromatic number*, or $\chi_B(G) \leq \aleph_0$, if there is a Borel $\mathbb{N}$-coloring of G. Our first result generalizes Kechris–Solecki–Todorcevic's characterization of the existence of such colorings (see [KST99, Theorem 6.3]):

Theorem 1. *Suppose that X is a Hausdorff space and $(G_{i,j})_{i,j \in \mathbb{N}}$ is an increasing-in-j sequence of analytic digraphs on X. Then exactly one of the following holds:*

1. *There are Borel sets $B_i \subseteq X$ with the property that $X = \bigcup_{i \in \mathbb{N}} B_i$ and $\forall i, j \in \mathbb{N}\ \chi_B(G_{i,j} \upharpoonright B_i) \leq \aleph_0$.*
2. *There exist a function $f \colon \mathbb{N} \to \mathbb{N}$ and a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \to X$ from $(\mathbb{G}_{0,k})_{k \in \mathbb{N}}$ to $(G_{k,f(k)})_{k \in \mathbb{N}}$.*

We use 1_Γ to denote the identity element of a group Γ. The *orbit equivalence relation* induced by a group action $\Gamma \curvearrowright X$ is the equivalence relation on X given by $x\ E_\Gamma^X\ y \iff \exists \gamma \in \Gamma\ \gamma \cdot x = y$. More generally, the *orbit relation* associated with a set $\Delta \subseteq \Gamma$ is the binary relation on X given by $x\ R_\Delta^X\ y \iff \exists \delta \in \Delta\ \delta \cdot x = y$. A set $Y \subseteq X$ is Δ-*lacunary* if $y\ R_\Delta^X\ z \implies y = z$ for all $y, z \in Y$, and E_Γ^X-*complete* if $X = \Gamma Y$.

We say that a Borel action $\Gamma \curvearrowright X$ of an analytic Hausdorff group on an analytic Hausdorff space is *(Borel) σ-lacunary* if there exist a sequence $(\Delta_n)_{n \in \mathbb{N}}$ of open neighborhoods of 1_Γ and a cover $(X_n)_{n \in \mathbb{N}}$ of X by E_Γ^X-invariant Borel sets with the property that there is a Δ_n-lacunary $E_\Gamma^{X_n}$-complete Borel set $B_n \subseteq X_n$ for all $n \in \mathbb{N}$.

A topological group is *tsi* if it has a compatible two-sided-invariant metric. Klee has shown that a Hausdorff group Γ is tsi if and only if there is a neighborhood basis of 1_Γ consisting of conjugation-invariant open sets (see [Kle52, 1.5]). A topological group is *cli* if it has a compatible complete left-invariant metric, or equivalently, a compatible complete right-invariant metric (see, for example, [Bec98, Proposition 3.A.2]). It is well known that every tsi group is cli (see, for example, [BK96, Corollary 1.2.2]). Our second result characterizes the class of σ-lacunary Borel actions of tsi Polish groups on Polish spaces:

Theorem 2. *Suppose that Γ is a cli Polish group, $(\Delta_k)_{k\in\mathbb{N}}$ is a neighborhood basis of 1_Γ, X is a Polish space, and $\Gamma \curvearrowright X$ is a Borel action with the property that R^X_Δ is Borel for all open sets $\Delta \subseteq \Gamma$. Then at least one of the following holds:*

1. *The action $\Gamma \curvearrowright X$ is σ-lacunary.*
2. *There exist a subsequence $(\Delta'_k)_{k\in\mathbb{N}}$ of $(\Delta_k)_{k\in\mathbb{N}}$ and a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $(\mathbb{G}_{0,k})_{k\in\mathbb{N}}$ to $(R^X_{\Delta'_k} \setminus R^X_{\Delta'_{k+1}})_{k\in\mathbb{N}}$.*

Moreover, if $(\Delta_k)_{k\in\mathbb{N}}$ is a decreasing sequence of conjugation-invariant sets, then exactly one of these conditions holds.

Following the usual abuse of language, we say that an equivalence relation E on X is *countable* if $|[x]_E| \leq \aleph_0$ for all $x \in X$. We use $=_X$ and $\neq_X$ to denote the equality and inequality relations on X, as well as $\mathbb{E}_0$ to denote the equivalence relation on $2^{\mathbb{N}}$ given by $c \mathrel{\mathbb{E}_0} d \iff \exists n \in \mathbb{N} \forall m \geq n\ c(m) = d(m)$. The *product* of equivalence relations E_n on X_n, for $n \in N$, is the equivalence relation $\prod_{n\in N} E_n$ on $\prod_{n\in N} X_n$ given by $(x_n)_{n\in N} \ (\prod_{n\in N} E_n)\ (y_n)_{n\in N} \iff \forall n \in N\ x_n \mathrel{E_n} y_n$. When $N = 2$, we use $E_0 \times E_1$ to denote the product. In the further special case that there exist $n \in \mathbb{N}$ and a set X for which $X_0 = X^n$ and $X_1 = X^{\mathbb{N}}$, we will abuse notation by identifying $E_0 \times E_1$ with the equivalence relation on $X^{\mathbb{N}}$ obtained via the obvious identification of $X^n \times X^{\mathbb{N}}$ with $X^{\mathbb{N}}$. The *N-fold power* of an equivalence relation E is given by $E^N = \prod_{n\in N} E$.

A *reduction* of a binary relation R to a binary relation S is a homomorphism from $(R, \sim R)$ to $(S, \sim S)$. An *embedding* of R into S is an injective reduction of R to S.

Given a Borel action $\Gamma \curvearrowright X$ of a Polish group on a Polish space for which E^X_Γ is Borel, we say that E^X_Γ is *essentially countable* if it is Borel reducible to a countable Borel equivalence relation on a Polish space. It is easy to see that if $\Gamma \curvearrowright X$ is σ-lacunary, then E^X_Γ is essentially countable (see Proposition 2.7). It is well known that $\mathbb{E}_0^{\mathbb{N}}$—which is clearly the orbit equivalence relation induced by a continuous action of the abelian Polish group $((\mathbb{Z}/2\mathbb{Z})^{<\mathbb{N}})^{\mathbb{N}}$—is not essentially countable (see the remarks preceding Proposition 3.8).

A topological group Γ is *non-archimedean* if there is a neighborhood basis of 1_Γ consisting of open subgroups. It follows that a Hausdorff group Γ is both non-archimedean and tsi if and only if there is a neighborhood basis of 1_Γ consisting of normal open subgroups (see, for example, [GX14, §2]). Our third result strengthens Hjorth–Kechris's theorem that if Γ is a non-archimedean tsi Polish group, X is a Polish space, $\Gamma \curvearrowright X$ is Borel, and E_Γ^X is Borel, then either E_Γ^X is essentially countable or there is a continuous embedding of $\mathbb{E}_0^{\mathbb{N}}$ into E_Γ^X (see [HK01, Theorem 8.1]):

Theorem 3. *Suppose that Γ is a non-archimedean tsi Polish group, $(\Gamma_k)_{k \in \mathbb{N}}$ is a sequence of open subgroups of Γ, X is a Polish space, $\Gamma \curvearrowright X$ is Borel, and E_Γ^X is Borel. Then exactly one of the following holds:*

1. *The action $\Gamma \curvearrowright X$ is σ-lacunary.*
2. *There is a continuous embedding $\phi \colon 2^{\mathbb{N} \times \mathbb{N}} \to X$ of $\mathbb{E}_0^{\mathbb{N}}$ into E_Γ^X that is a homomorphism from $((=_{2^{\mathbb{N}}})^k \times \mathbb{E}_0^{\mathbb{N}})_{k \in \mathbb{N}}$ to $(E_{\Gamma_k}^X)_{k \in \mathbb{N}}$.*

In §1, we establish Theorem 1. In §2, we consider the connection between σ-lacunarity and condition (1) of Theorem 1. In §3, we describe ways of refining condition (2) of Theorem 1. And in §4, we establish Theorems 2 and 3.

1. A generalization of the $\mathbb{G}_0$ dichotomy

A set Z *separates* a set X from a set Y if $X \subseteq Z$ and $Y \cap Z = \emptyset$. Given sets $X' \subseteq X$, $Y' \subseteq Y$, and $R \subseteq X \times Y$, we say that (X', Y') is *R-independent* if $R \cap (X' \times Y') = \emptyset$.

Proposition 1.1. *Suppose that X and Y are Hausdorff spaces, $A_X \subseteq X$, $A_Y \subseteq Y$, and $R \subseteq X \times Y$ are analytic, and (A_X, A_Y) is R-independent. Then there are Borel sets $B_X \subseteq X$ and $B_Y \subseteq Y$ for which $A_X \subseteq B_X$, $A_Y \subseteq B_Y$, and (B_X, B_Y) is R-independent.*

Proof. As the R-independence of (A_X, A_Y) ensures that A_X is disjoint from $\mathrm{proj}_X((X \times A_Y) \cap R)$, and the latter set can be expressed as $\mathrm{proj}_X((\mathrm{proj}_X(R) \times A_Y) \cap R)$—and is therefore analytic (see, for example, the proof of [Kec95, Proposition 14.4])—Lusin's separation theorem (see, for example, the proof of [Kec95, Theorem 14.7]) yields a Borel set $B_X \subseteq X$ separating A_X from $\mathrm{proj}_X((X \times A_Y) \cap R)$. Then (B_X, A_Y) is R-independent, so A_Y is disjoint from $\mathrm{proj}_Y((B_X \times Y) \cap R)$, and since the latter set is analytic, another application of Lusin's separation theorem yields a Borel set $B_Y \subseteq Y$ separating A_Y from $\mathrm{proj}_Y((B_X \times Y) \cap R)$, in which case (B_X, B_Y) is R-independent. ⊠

Proposition 1.2. *Suppose that X is a Hausdorff space, G is an analytic digraph on X, and $A \subseteq X$ is a G-independent analytic set. Then there is a G-independent Borel set $B \subseteq X$ for which $A \subseteq B$.*

Proof. By Proposition 1.1, there are Borel sets $B_i \subseteq X$ such that $A \subseteq B_i$ for all $i < 2$ and $(B_i)_{i<2}$ is G-independent. Set $B = \bigcap_{i<2} B_i$. ⊠

Given sets $M \subseteq N$ and a sequence $s \in 2^M$, let $\mathcal{N}_s$ denote the set of sequences $c \in 2^N$ for which $s \sqsubseteq c$.

Proof of Theorem 1. To see that conditions (1) and (2) are mutually exclusive, suppose that both hold, and fix $i \in \mathbb{N}$ for which $\phi^{-1}(B_i)$ is non-meager, as well as a Borel coloring $\pi \colon B_i \to \mathbb{N}$ of $G_{i,f(i)} \restriction B_i$. Then there exists $m \in \mathbb{N}$ for which the set $C = (\pi \circ \phi)^{-1}(\{m\})$ is non-meager. Fix $s \in 2^{<\mathbb{N}}$ such that C is comeager in $\mathcal{N}_s$ (see, for example, [Kec95, Proposition 8.26]), as well as $n \in \mathbb{N}$ for which $i = k_n$ and $s \sqsubseteq s_n$. Define $\iota \colon \mathcal{N}_{s_n \frown (0)} \to \mathcal{N}_{s_n \frown (1)}$ by $\iota(s_n \frown (0) \frown c) = s_n \frown (1) \frown c$, for all $c \in 2^{\mathbb{N}}$. As ι is a homeomorphism, the set $C \cap \iota^{-1}(C)$ is comeager in $\mathcal{N}_{s_n \frown (0)}$ (see, for example, [Kec95, Exercise 8.45]). But if $c \in C \cap \iota^{-1}(C)$, then $\phi(c)\ (G_{i,f(i)} \restriction B_i)\ (\phi \circ \iota)(c)$ and $(\pi \circ \phi)(c) = (\pi \circ \phi \circ \iota)(c) = m$, contradicting the fact that π is a coloring of $G_{i,f(i)} \restriction B_i$.

It remains to show that at least one of conditions (1) and (2) holds. We can assume that, for all $i \in \mathbb{N}$, there exists $j \in \mathbb{N}$ with the property that $G_{i,j} \neq \emptyset$, since otherwise condition (1) holds trivially. By removing a finite initial segment of $(G_{i,j})_{j \in \mathbb{N}}$ for all $i \in \mathbb{N}$, we can therefore assume that $G_{i,j} \neq \emptyset$ for all $i, j \in \mathbb{N}$, in which case there are continuous surjections $\psi_{i,j} \colon \mathbb{N}^{\mathbb{N}} \to G_{i,j}$, for all $i, j \in \mathbb{N}$. Letting proj_k denote projection onto the k^{th} coordinate, it similarly follows that there is a continuous surjection $\phi_X \colon \mathbb{N}^{\mathbb{N}} \to \bigcup_{i,j \in \mathbb{N}, k<2} \mathrm{proj}_k(G_{i,j})$.

We will recursively define decreasing sequences $(X^\alpha_{i,j})_{\alpha<\omega_1}$ of Borel subsets of X such that $X^\alpha_{i,j} \subseteq X^\alpha_{i,j+1}$ and $\chi_B(G_{i,j} \restriction {\sim}X^\alpha_{i,j}) \leq \aleph_0$ for all $\alpha < \omega_1$ and $i, j \in \mathbb{N}$, so $\chi_B(G_{i,j} \restriction {\sim}\bigcup_{k \in \mathbb{N}} X^\alpha_{i,k}) \leq \aleph_0$ for all $i, j \in \mathbb{N}$, thus condition (1) holds if and only if it holds on $\bigcap_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} X^\alpha_{i,j}$. We begin by setting $X^0_{i,j} = X$ for all $i, j \in \mathbb{N}$. We define $X^\lambda_{i,j} = \bigcap_{\alpha<\lambda} X^\alpha_{i,j}$ for all $i, j \in \mathbb{N}$ and limit ordinals $\lambda < \omega_1$. To describe the construction of $X^{\alpha+1}_{i,j}$ from $X^\alpha_{i,j}$, we require several preliminaries.

We say that a quadruple $a = (n^a, f^a, \phi^a, (\psi^a_n)_{n<n^a})$ is an *approximation* if $n^a \in \mathbb{N}$, $f^a \colon \{k_n \mid n < n^a\} \to \mathbb{N}$, $\phi^a \colon 2^{n^a} \to \mathbb{N}^{n^a}$, and $\psi^a_n \colon 2^{n^a-1-n} \to \mathbb{N}^{n^a}$ for all $n < n^a$. We say that an approximation b is a *one-step extension* of an approximation a if:

- $n^a = n^b - 1$.
- $f^a = f^b \restriction \{k_n \mid n < n^a\}$.
- $\forall i < 2 \forall s \in 2^{n^a}\ \phi^a(s) \sqsubseteq \phi^b(s \frown (i))$.
- $\forall i < 2 \forall n < n^a \forall s \in 2^{n^a-n-1}\ \psi^a_n(s) \sqsubseteq \psi^b_n(s \frown (i))$.

We say that a quadruple $\gamma = (n^\gamma, f^\gamma, \phi^\gamma, (\psi^\gamma_n)_{n<n^\gamma})$ is a *configuration* if $n^\gamma \in \mathbb{N}$, $f^\gamma \colon \{k_n \mid n < n^\gamma\} \to \mathbb{N}$, $\phi^\gamma \colon 2^{n^\gamma} \to \mathbb{N}^{\mathbb{N}}$, $\psi^\gamma_n \colon 2^{n^\gamma-1-n} \to \mathbb{N}^{\mathbb{N}}$ for all $n < n^\gamma$, and $(\psi_{k_n, f^\gamma(k_n)} \circ \psi^\gamma_n)(s) = ((\phi_X \circ \phi^\gamma)(s_n \frown (i) \frown s))_{i<2}$ for all $n < n^\gamma$ and $s \in 2^{n^\gamma - n - 1}$. We say that a configuration γ is *compatible* with an approximation a if:

- $n^a = n^\gamma$.
- $f^a = f^\gamma$.
- $\forall s \in 2^{n^a}\ \phi^a(s) \sqsubseteq \phi^\gamma(s)$.
- $\forall n < n^a \forall s \in 2^{n^a - n - 1}\ \psi^a_n(s) \sqsubseteq \psi^\gamma_n(s)$.

We say that a configuration γ is *compatible* with a sequence $(X_{i,j})_{i,j \in \mathbb{N}}$ of subsets of X if there is an extension $f \colon \mathbb{N} \to \mathbb{N}$ of f^γ with the property that $(\phi_X \circ \phi^\gamma)(2^{n^\gamma}) \subseteq \bigcap_{i \in \mathbb{N}} X_{i,f(i)}$. We say that an approximation a is $(X_{i,j})_{i,j \in \mathbb{N}}$-*terminal* if no configuration is compatible with both a one-step extension of a and $(X_{i,j})_{i,j \in \mathbb{N}}$. Let $A(a, (X_{i,j})_{i,j \in \mathbb{N}})$ denote the set of points of the form $(\phi_X \circ \phi^\gamma)(s_{n^a})$, where γ varies over configurations compatible with both a and $(X_{i,j})_{i,j \in \mathbb{N}}$. Note that if $(X_{i,j})_{i,j \in \mathbb{N}}$ is a sequence of Borel sets, then $A(a, (X_{i,j})_{i,j \in \mathbb{N}})$ is a continuous image of a Borel subset of $\mathbb{N}^\mathbb{N}$, and is therefore analytic.

Lemma 1.3. *Suppose that $(X_{i,j})_{i,j \in \mathbb{N}}$ is a sequence of subsets of X and a is an approximation for which $k_{n^a} \in \mathrm{dom}(f^a)$ and $A(a, (X_{i,j})_{i,j \in \mathbb{N}})$ is not $G_{k_{n^a}, f^a(k_{n^a})}$-independent. Then a is not $(X_{i,j})_{i,j \in \mathbb{N}}$-terminal.*

Proof. Fix configurations γ_0 and γ_1, compatible with a and $(X_{i,j})_{i,j \in \mathbb{N}}$, for which $((\phi_X \circ \phi^{\gamma_i})(s_{n^a}))_{i<2} \in G_{k_{n^a}, f^a(k_{n^a})}$. Then there exists $b \in \mathbb{N}^\mathbb{N}$ such that $\psi_{k_{n^a}, f^a(k_{n^a})}(b) = ((\phi_X \circ \phi^{\gamma_i})(s_{n^a}))_{i<2}$. Let γ be the configuration given by $n^\gamma = n^a + 1$, $f^\gamma = f^a$, $\phi^\gamma(s \frown (i)) = \phi^{\gamma_i}(s)$ for all $i < 2$ and $s \in 2^{n^a}$, $\psi^\gamma_n(s \frown (i)) = \psi^{\gamma_i}_n(s)$ for all $i < 2$, $n < n^a$, and $s \in 2^{n^a - n - 1}$, and $\psi^\gamma_{n^a}(\emptyset) = b$. Then the unique approximation with which γ is compatible is a one-step extension of a, so a is not $(X_{i,j})_{i,j \in \mathbb{N}}$-terminal. ⊠

Lemma 1.4. *Suppose that $(X_{i,j})_{i,j \in \mathbb{N}}$ is a sequence of subsets of X, a is an approximation for which $k_{n^a} \notin \mathrm{dom}(f^a)$, and there exists $\ell \in \mathbb{N}$ such that $A(a, (X_{i,j})_{i,j \in \mathbb{N}})$ is not $G_{k_{n^a}, \ell}$-independent. Then a is not $(X_{i,j})_{i,j \in \mathbb{N}}$-terminal.*

Proof. Fix configurations γ_0 and γ_1, compatible with a and $(X_{i,j})_{i,j \in \mathbb{N}}$, for which $((\phi_X \circ \phi^{\gamma_i})(s_{n^a}))_{i<2} \in G_{k_{n^a}, \ell}$. By increasing ℓ if necessary, we can assume that $(\phi_X \circ \phi^{\gamma_0})(2^{n^a}) \cup (\phi_X \circ \phi^{\gamma_1})(2^{n^a}) \subseteq X_{k_{n^a}, \ell}$. Fix $b \in \mathbb{N}^\mathbb{N}$ such that $\psi_{k_{n^a}, \ell}(b) = ((\phi_X \circ \phi^{\gamma_i})(s_{n^a}))_{i<2}$, and let γ be the configuration given by $n^\gamma = n^a + 1$, $f^\gamma(k) = f^a(k)$ for all $k < k_{n^a}$, $f^\gamma(k_{n^a}) = \ell$, $\phi^\gamma(s \frown (i)) = \phi^{\gamma_i}(s)$ for all $i < 2$ and $s \in 2^{n^a}$, $\psi^\gamma_n(s \frown (i)) = \psi^{\gamma_i}_n(s)$ for all $i < 2$, $n < n^a$, and $s \in 2^{n^a - n - 1}$, and $\psi^\gamma_{n^a}(\emptyset) = b$. Then the unique approximation with which γ is compatible is a one-step extension of a, so a is not $(X_{i,j})_{i,j \in \mathbb{N}}$-terminal. ⊠

As Proposition 1.2 ensures that every $G_{i,j}$-independent analytic set is contained in a $G_{i,j}$-independent Borel set, Lemmas 1.3 and 1.4 imply that if $(X_{i,j})_{i,j\in\mathbb{N}}$ is a sequence of Borel sets and a is an $(X_{i.j})_{i,j\in\mathbb{N}}$-terminal approximation, then there is a Borel set $B(a,(X_{i,j})_{i,j\in\mathbb{N}}) \supseteq A(a,(X_{i,j})_{i,j\in\mathbb{N}})$ that is $G_{k_{n^a},f^a(k_{n^a})}$-independent if $k_{n^a} \in \mathrm{dom}(f^a)$, and $G_{k_{n^a},\ell}$-independent for all $\ell \in \mathbb{N}$ if $k_{n^a} \notin \mathrm{dom}(f^a)$.

We finally define $X^{\alpha+1}_{k,\ell}$ to be the difference of $X^\alpha_{k,\ell}$ and the union of the sets of the form $B(a,(X^\alpha_{i,j})_{i,j\in\mathbb{N}})$, where a is an $(X^\alpha_{i,j})_{i,j\in\mathbb{N}}$-terminal approximation, $k_{n^a} = k$, and $f^a(k_{n^a}) \geq \ell$ if $k_{n^a} \in \mathrm{dom}(f^a)$.

Lemma 1.5. *Suppose that $\alpha < \omega_1$ and a is an approximation that is not $(X^{\alpha+1}_{i,j})_{i,j\in\mathbb{N}}$-terminal. Then there is a one-step extension of a that is not $(X^\alpha_{i,j})_{i,j\in\mathbb{N}}$-terminal.*

Proof. Fix a one-step extension b of a for which there is a configuration γ compatible with b and $(X^{\alpha+1}_{i,j})_{i,j\in\mathbb{N}}$. Note that if $k_{n^b} \in \mathrm{dom}(f^b)$, then $(\phi_X \circ \phi^\gamma)(s_{n^b}) \in X^{\alpha+1}_{k_{n^b},f^b(k_{n^b})}$, so $A(b,(X^\alpha_{i,j})_{i,j\in\mathbb{N}}) \cap X^{\alpha+1}_{k_{n^b},f^b(k_{n^b})} \neq \emptyset$, thus b is not $(X^\alpha_{i,j})_{i,j\in\mathbb{N}}$-terminal. And if $k_{n^b} \notin \mathrm{dom}(f^b)$, then there exists $\ell \in \mathbb{N}$ for which $(\phi_X \circ \phi^\gamma)(s_{n^b}) \in X^{\alpha+1}_{k_{n^b},\ell}$, so $A(b,(X^\alpha_{i,j})_{i,j\in\mathbb{N}}) \cap X^{\alpha+1}_{k_{n^b},\ell} \neq \emptyset$, thus b is not $(X^\alpha_{i,j})_{i,j\in\mathbb{N}}$-terminal. ⊠

Fix $\alpha < \omega_1$ for which the families of $(X^\alpha_{i,j})_{i,j\in\mathbb{N}}$- and $(X^{\alpha+1}_{i,j})_{i,j\in\mathbb{N}}$-terminal approximations are one and the same, and let a_0 be the unique approximation such that $n^{a_0} = 0$. Then $A(a_0,(X_{i,j})_{i,j\in\mathbb{N}}) = \phi_X(\mathbb{N}^\mathbb{N}) \cap \bigcap_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} X_{i,j}$ for all sequences $(X_{i,j})_{i,j\in\mathbb{N}}$ of subsets of X, so if a_0 is $(X^\alpha_{i,j})_{i,j\in\mathbb{N}}$-terminal, then $X^{\alpha+1}_{0,\ell} \subseteq X^\alpha_{0,\ell} \setminus (\phi_X(\mathbb{N}^\mathbb{N}) \cap \bigcap_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} X^\alpha_{i,j})$ for all $\ell \in \mathbb{N}$, so $\bigcap_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} X^{\alpha+1}_{i,j} \subseteq (\bigcup_{\ell\in\mathbb{N}} X^\alpha_{0,\ell} \setminus (\phi_X(\mathbb{N}^\mathbb{N}) \cap \bigcap_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} X^\alpha_{i,j})) \cap \bigcap_{i>0} \bigcup_{j\in\mathbb{N}} X^\alpha_{i,j}$. As the latter set is disjoint from $\phi_X(\mathbb{N}^\mathbb{N})$, it follows that condition (1) holds on $\bigcap_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} X^{\alpha+1}_{i,j}$, thus condition (1) holds.

Otherwise, by recursively applying Lemma 1.5, we obtain one-step extensions a_{n+1} of a_n that are not $(X^\alpha_{i,j})_{i,j\in\mathbb{N}}$-terminal, for all $n \in \mathbb{N}$. Define $f \colon \mathbb{N} \to \mathbb{N}$ by $f = \bigcup_{n\in\mathbb{N}} f^{a_n}$, define $\phi \colon 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\phi(c) = \bigcup_{n\in\mathbb{N}} \phi^{a_n}(c \upharpoonright n)$ for all $c \in 2^\mathbb{N}$, and define $\psi_n \colon 2^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\psi_n(c) = \bigcup_{m\in\mathbb{N}} \psi_n^{a_{n+1+m}}(c \upharpoonright m)$ for all $c \in 2^\mathbb{N}$ and $n \in \mathbb{N}$. To see that $\phi_X \circ \phi$ is a homomorphism from $(\mathbb{G}_{0,k})_{k\in\mathbb{N}}$ to $(G_{k,f(k)})_{k\in\mathbb{N}}$, we will show that $(\psi_{k_n,f(k_n)} \circ \psi_n)(c) = ((\phi_X \circ \phi)(s_n \frown (i) \frown c))_{i<2}$ for all $c \in 2^\mathbb{N}$ and $n \in \mathbb{N}$. For this, it is sufficient to show that if $U \subseteq X \times X$ is an open neighborhood of $(\psi_{k_n,f(k_n)} \circ \psi_n)(c)$ and $V \subseteq X \times X$ is an open neighborhood of $((\phi_X \circ \phi)(s_n \frown (i) \frown c))_{i<2}$, then $U \cap V \neq \emptyset$. Towards this end, fix $m \in \mathbb{N}$ for which $\psi_{k_n,f(k_n)}(\mathcal{N}_{\psi_n^{a_{n+1+m}}(s)}) \subseteq U$ and $\prod_{i<2} \phi_X(\mathcal{N}_{\phi^{a_{n+1+m}}(s_n \frown (i) \frown s)}) \subseteq V$, where $s = c \upharpoonright m$. As a_{n+1+m} is not

$(X^\alpha_{i,j})_{i,j \in \mathbb{N}}$-terminal, there is a configuration γ compatible with a_{n+1+m}. Then $(\psi_{k_n,f(k_n)} \circ \psi^\gamma_n)(s) \in U$ and $((\phi_X \circ \phi^\gamma)(s_n \frown (i) \frown s))_{i<2} \in V$, thus $U \cap V \neq \emptyset$. ⊠

2. Lacunary sets

We provide the proof of the following straightforward observation for the reader's convenience:

Proposition 2.1. *Suppose that Γ is a topological group, $D \subseteq \Gamma$ is dense, and $U \subseteq \Gamma$ is a non-empty open set. Then $\Gamma = DU = UD$.*

Proof. Note that if $\gamma \in \Gamma$, then γU^{-1} and $U^{-1}\gamma$ are non-empty and open, so $D \cap \gamma U^{-1}$ and $U^{-1}\gamma \cap D$ are non-empty, and it follows that $\gamma \in (D \cap \gamma U^{-1})U \subseteq DU$ and $\gamma \in U(D \cap U^{-1}\gamma) \subseteq UD$. ⊠

We next show that σ-lacunarity yields the corresponding special case of condition (1) in Theorem 1:

Proposition 2.2. *Suppose that Γ is an analytic Hausdorff group, $(\Delta_i)_{i \in \mathbb{N}}$ is a neighborhood basis of 1_Γ consisting of conjugation-invariant open sets, X is an analytic Hausdorff space, and $\Gamma \curvearrowright X$ is a σ-lacunary Borel action with the property that R^X_Δ is Borel for all open sets $\Delta \subseteq \Gamma$. Then there are Borel sets $B_i \subseteq X$ with the property that $X = \bigcup_{i \in \mathbb{N}} B_i$ and $\forall i, j \in \mathbb{N}\ \chi_B((R^X_{\Delta_i} \setminus R^X_{\Delta_j}) \restriction B_i) \leq \aleph_0$.*

Proof. By breaking X into countably-many E^X_Γ-invariant Borel sets, we can assume that there is an open neighborhood $\Delta \subseteq \Gamma$ of 1_Γ for which there is a Δ-lacunary E^X_Γ-complete Borel set $B \subseteq X$.

Fix $i \in \mathbb{N}$ for which there is an open neighborhood $\Delta' \subseteq \Gamma$ of 1_Γ such that $(\Delta')^{-1}\Delta_i\Delta' \subseteq \Delta$. To see that $\forall j \in \mathbb{N}\ \chi_B(R^X_{\Delta_i} \setminus R^X_{\Delta_j}) \leq \aleph_0$, suppose that $j \in \mathbb{N}$, and fix a non-empty open set $\Delta'' \subseteq \Delta'$ for which $\Delta''(\Delta'')^{-1} \subseteq \Delta_j$.

Lemma 2.3. *The set $\Delta'' B$ is $(R^X_{\Delta_i} \setminus R^X_{\Delta_j})$-independent.*

Proof. Suppose that $x'', y'' \in \Delta'' B$ are $R^X_{\Delta_i}$-related, and fix $\delta''_x, \delta''_y \in \Delta''$ for which the points $x = (\delta''_x)^{-1} \cdot x''$ and $y = (\delta''_y)^{-1} \cdot y''$ are in B. Then x and y are $R^X_{(\Delta'')^{-1}\Delta_i\Delta''}$-related, so R^X_Δ-related, thus equal, and it follows that x'' and y'' are $R^X_{\Delta''(\Delta'')^{-1}}$-related, thus $R^X_{\Delta_j}$-related. ⊠

For all $\gamma \in \Gamma$, Lemma 2.3 and the conjugation invariance of Δ_i and Δ_j ensure that $\gamma\Delta'' B$ is $(R^X_{\Delta_i} \setminus R^X_{\Delta_j})$-independent. As $\gamma\Delta'' B$ is analytic, Proposition 1.2 therefore yields an $(R^X_{\Delta_i} \setminus R^X_{\Delta_j})$-independent Borel set $B_\gamma \subseteq X$ containing $\gamma\Delta'' B$.

Fix a countable dense set $D \subseteq \Gamma$. As Proposition 2.1 ensures that $\Gamma = D\Delta''$, it follows that $X = \Gamma B = \bigcup_{\gamma \in D} \gamma\Delta'' B = \bigcup_{\gamma \in D} B_\gamma$, thus $\chi_B(R^X_{\Delta_i} \setminus R^X_{\Delta_j}) \leq \aleph_0$. ⊠

Given a digraph G on a set X, we say that a set $Y \subseteq X$ is a *G-clique* if all pairs of distinct points of Y are in G.

Proposition 2.4. *Suppose that Γ is a separable group, X is a set, $\Gamma \curvearrowright X$ is an action, and $\Delta \subseteq \Gamma$ is an open neighborhood of 1_Γ. Then every E^X_Γ-class is a countable union of $(R^X_\Delta \setminus =_X)$-cliques.*

Proof. Fix a countable dense set $D \subseteq \Gamma$ and a non-empty open set $\Lambda \subseteq \Gamma$ for which $\Lambda\Lambda^{-1} \subseteq \Delta$. Observe that if $d \in D$ and $x \in X$, then $y, z \in \Lambda d \cdot x \implies z \in \Lambda d(\Lambda d)^{-1} y = \Lambda\Lambda^{-1} y \subseteq \Delta y$, so $\Lambda d \cdot x$ is an $(R^X_\Delta \setminus =_X)$-clique, and $\Gamma x = \bigcup_{d \in D} \Lambda d \cdot x$ by Proposition 2.1. ⊠

We next show that σ-lacunarity follows from the corresponding special case of condition (1) in Theorem 1:

Proposition 2.5. *Suppose that Γ is a cli Polish group, X is an analytic metric space, $\Gamma \curvearrowright X$ is continuous, $(\Delta_i)_{i \in \mathbb{N}}$ is a neighborhood basis of 1_Γ, and there are Borel sets $B_i \subseteq X$ with the property that $X = \bigcup_{i \in \mathbb{N}} B_i$ and $\forall i, j \in \mathbb{N}\ \chi_B((R^X_{\Delta_i} \setminus R^X_{\Delta_j}) \restriction B_i) \leq \aleph_0$. Then $\Gamma \curvearrowright X$ is σ-lacunary.*

Proof. We can assume that Γ is not discrete, since otherwise $\Gamma \curvearrowright X$ is trivially σ-lacunary. So, by passing to a subsequence of $(\Delta_i)_{i \in \mathbb{N}}$, we can assume that $(\overline{\Delta_{i+1}})^{-1} \cup (\overline{\Delta_{i+1}})^2 \subseteq \Delta_i$ for all $i \in \mathbb{N}$. By breaking each B_i into countably-many Borel sets, we obtain Borel sets $B'_n \subseteq X$ and $i_n \in \mathbb{N}$ with the property that B'_n is $(R^X_{\Delta_{i_n}} \setminus R^X_{\Delta_{i_n+3}})$-independent and $\chi_B((R^X_{\Delta_{i_n}} \setminus R^X_{\Delta_j}) \restriction B'_n) \leq \aleph_0$ for all $j \geq i_n + 4$ and $n \in \mathbb{N}$. As a result of Montgomery-Novikov ensures that the class of Borel sets is closed under category quantification (see, for example, [Kec95, Theorem 16.1]), it follows that the function $\phi \colon X \to \mathbb{N}$, given by $\phi(x) = \min\{n \in \mathbb{N} \mid \exists^* \gamma \in \Gamma\ \gamma \cdot x \in B'_n\}$, is Borel. By passing to the E^X_Γ-invariant Borel sets of the form $\phi^{-1}(\{n\})$, where $n \in \mathbb{N}$, it is sufficient to show that if $i \in \mathbb{N}$ and $B \subseteq X$ is an $(R^X_{\Delta_i} \setminus R^X_{\Delta_{i+3}})$-independent Borel set such that $\forall j \geq i + 4\ \chi_B((R^X_{\Delta_i} \setminus R^X_{\Delta_j}) \restriction B) \leq \aleph_0$ and $\forall x \in X \exists^* \gamma \in \Gamma\ \gamma \cdot x \in B$, then there is a Δ_{i+2}-lacunary E^X_Γ-complete Borel set.

Lemma 2.6. *The restriction $E = R^X_{\Delta_i} \restriction B$ is an equivalence relation.*

Proof. To see that E is symmetric, observe that if $x\ E\ y$, then the $(R^X_{\Delta_i} \setminus R^X_{\Delta_{i+3}})$-independence of B ensures that $x\ R^X_{\Delta_{i+3}}\ y$, so $y\ R^X_{\Delta_{i+2}}\ x$, thus $y\ E\ x$. To see that E is transitive, note that if $x\ E\ y\ E\ z$, then the $(R^X_{\Delta_i} \setminus R^X_{\Delta_{i+3}})$-independence of B ensures that $x\ R^X_{\Delta_{i+3}}\ y\ R^X_{\Delta_{i+3}}\ z$, so $x\ R^X_{\Delta_{i+2}}\ z$, thus $x\ E\ z$. ⊠

Proposition 2.4 ensures that E has countable index below $E^X_\Gamma \upharpoonright B$, so the set $B' = \{x \in B \mid \exists^* \gamma \in \Gamma \; x \; E \; \gamma \cdot x\}$ is E^X_Γ-complete. By replacing B with B', we can therefore assume that $\forall x \in B \exists^* \gamma \in \Gamma \; x \; E \; \gamma \cdot x$.

Fix positive real numbers $\epsilon_j \to 0$ and Borel colorings $c_j \colon B \to \mathbb{N}$ of $(R^X_{\Delta_i} \setminus R^X_{\Delta_j}) \upharpoonright B$ such that $\operatorname{diam} c_j^{-1}(\{n\}) \leq \epsilon_j$ for all $j \geq i + 4$ and $n \in \mathbb{N}$. For all $j \geq i + 3$ and $x \in B$, let $s_j(x)$ be the lexicographically minimal sequence $s \in \mathbb{N}^{j-i-3}$ for which there are non-meagerly many $\gamma \in \Gamma$ such that $\gamma \cdot x \in [x]_E \cap \bigcap_{i+4 \leq k \leq j} c_k^{-1}(\{s(k - i - 4)\})$, and define $C_j = \{x \in B \mid s_j(x) = (c_k(x))_{i+4 \leq k \leq j}\}$.

A *ray* from a point $x \in B$ through $(C_j)_{j \geq i+3}$ is a sequence $(\delta_j)_{j \geq i+3}$ such that $\delta_j \in \Delta_j$ and $\delta_j \cdots \delta_{i+3} \cdot x \in C_{j+1}$ for all $j \geq i+3$. To see that such rays exist, set $x_{i+3} = x$, and fix $x_j \in C_j \cap [x]_E$ for all $j \geq i + 4$. As the $(R^X_{\Delta_i} \setminus R^X_{\Delta_j})$-independence of C_j ensures that $E \upharpoonright C_j \subseteq R^X_{\Delta_j}$, there exists $\delta_j \in \Delta_j$ such that $\delta_j \cdot x_j = x_{j+1}$, for all $j \geq i+3$. But then $(\delta_j)_{j \geq i+3}$ is a ray from x through $(C_j)_{j \geq i+3}$.

As $\Delta_j^2 \subseteq \Delta_{j-1}$ for all $j \geq i+3$, a straightforward induction shows that if $i + 3 \leq j \leq k$, then $\Delta_k \cdots \Delta_j \subseteq \Delta_{j-1}$. It follows that if $(\delta_j)_{j \geq i+3}$ is a ray from x through $(C_j)_{j \geq i+3}$, then $\delta_k \cdots \delta_j \in \Delta_{j-1}$ for all $k \geq j \geq i+3$, so $(\delta_j \cdots \delta_{i+3})_{j \geq i+3}$ is Cauchy with respect to every compatible complete right-invariant metric on Γ, and therefore converges to some $\delta \in \overline{\Delta_{i+2}}$.

Observe now that if $(\delta^x_j)_{j \geq i+3}$ and $(\delta^y_j)_{j \geq 3}$ are rays from points x and y in B through the sequence $(C_j)_{j \geq i+3}$, and δ^x and δ^y are the corresponding limit points, then $(\delta^y)^{-1} \overline{\Delta_{i+2}} \delta^x \subseteq (\delta^y)^{-1} \Delta_{i+1} \subseteq \Delta_i$, so $\delta^x \cdot x \; R^X_{\Delta_{i+2}} \; \delta^y \cdot y \implies x \; R^X_{\Delta_i} \; y \implies x \; E \; y$, and the fact that $\operatorname{diam} C_j \to 0$ ensures that $x \; E \; y \implies \delta^x \cdot x = \delta^y \cdot y$. Define $\psi \colon B \to X$ by setting $\psi(x) = y$ if and only if there is a ray $(\delta_j)_{j \geq i+3}$ from x through $(C_j)_{j \geq i+3}$ for which $\delta_j \cdots \delta_{i+3} \cdot x \to y$. As $\operatorname{graph}(\psi)$ is analytic, it is Borel (see, for example, the proof of [Kec95, Theorem 14.12]), so the fact that $\psi(B) = \{x \in X \mid \exists^* \gamma \in \Gamma \; \psi(\gamma \cdot x) = x\}$ ensures that $\psi(B)$ is Borel. But the above remarks also imply that the latter set is Δ_{i+2}-lacunary, and it is clearly E^X_Γ-complete. ⊠

We close this section by noting that σ-lacunarity yields essential countability. By the Lusin-Novikov uniformization theorem (see, for example, [Kec95, Theorem 18.10]), it is sufficient to show the following:

Proposition 2.7. *Suppose that Γ is a Polish group, X is a Polish space, and $\Gamma \curvearrowright X$ is a σ-lacunary Borel action. Then there is an E^X_Γ-complete Borel set $B \subseteq X$ such that $E^X_\Gamma \upharpoonright B$ is countable.*

Proof. It is sufficient to show that if $\Delta \subseteq \Gamma$ is an open neighborhood of 1_Γ and $Y \subseteq X$ is Δ-lacunary, then $E^X_\Gamma \upharpoonright Y$ is countable. But this is a direct consequence of Proposition 2.4. ⊠

Remark 2.8. Although we shall have no need for it here, it is worth mentioning that—after seeing a draft of this article and having several conversations with myself on the topic—Grebík established the converse of Proposition 2.7 in the special case that E_Γ^X is Borel. While this can be combined with [HK01, Theorem 8.1] to obtain a version of Theorem 3, one of the primary motivations underlying this paper was, in fact, to provide a simpler proof of [HK01, Theorem 8.1]. The idea behind Grebík's argument is as follows: Appeal to the Lusin-Novikov uniformization theorem to obtain a Borel uniformization $\pi \colon X \to B$ of E_Γ^X, note that for all open neighborhoods $\Delta \subseteq \Gamma$ of 1_Γ, the sets $X_\Delta = \{x \in X \mid \forall^* \delta \in \Delta \ \pi(x) = \pi(\delta \cdot x)\}$ and $B_\Delta = \{y \in B \mid \exists^* \gamma \in \Gamma \ (\gamma \cdot y \in X_\Delta \text{ and } \pi(\gamma \cdot y) = y)\}$ are Borel, fix a Borel uniformization $\pi_\Delta \colon B_\Delta \to X_\Delta$ of $\{(y, x) \in B_\Delta \times X_\Delta \mid \pi(x) = y\}$ (see, for example, [Kec95, Theorem 18.6]), and observe that the set $A_\Delta = \pi_\Delta(B_\Delta)$ is Borel (see, for example, [Kec95, Theorem 15.1]). But one can easily check that A_Δ is Λ-lacunary for every open set $\Lambda \subseteq \Gamma$ such that $\Lambda^2 \subseteq \Delta$, and $\bigcup\{A_\Delta \mid \Delta \text{ is an open neighborhood of } 1_\Gamma\}$ is E_Γ^X-complete (see [Gre20]).

3. Compositions

Given $n \in \mathbb{N}$ and a sequence $(s_i)_{i<n}$ of elements of $\mathbb{N}^{<\mathbb{N}}$, let $\bigoplus_{i<n} s_i$ denote the concatenation $s_0 \frown s_1 \frown \cdots \frown s_{n-1}$. Given a sequence $(s_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{N}^{<\mathbb{N}}$, let $\bigoplus_{n \in \mathbb{N}} s_n$ denote $\bigcup_{n \in \mathbb{N}} \bigoplus_{i<n} s_i$.

Proposition 3.1. *Suppose that $C \subseteq 2^\mathbb{N}$ is comeager and $f \colon \mathbb{N} \to \mathbb{N}$. Then there is a continuous homomorphism $\phi \colon 2^\mathbb{N} \to C$ from $(\mathbb{G}_{0,k})_{k \in \mathbb{N}}$ to $(\mathbb{G}_{0,f(k)})_{k \in \mathbb{N}}$.*

Proof. Fix dense open sets $U_n \subseteq 2^\mathbb{N}$ for which $\bigcap_{n \in \mathbb{N}} U_n \subseteq C$.

Lemma 3.2. *For all $n \in \mathbb{N}$ and $\phi \colon 2^n \to 2^{<\mathbb{N}}$, there exists $t \in 2^{<\mathbb{N}}$ such that $\mathcal{N}_{\phi(s) \frown t} \subseteq U_n$ for all $s \in 2^n$.*

Proof. Fix an enumeration $(s_m)_{m<2^n}$ of 2^n, and recursively find $t_m \in 2^{<\mathbb{N}}$ with $\mathcal{N}_{\phi(s_m) \frown \bigoplus_{\ell \leq m} t_\ell} \subseteq U_n$ for all $m < 2^n$. Set $t = \bigoplus_{m<2^n} t_m$. ⊠

Set $\ell_0 = 0$ and define $\phi_0 \colon 2^0 \to 2^{\ell_0}$ by $\phi_0(\emptyset) = \emptyset$. Given $n \in \mathbb{N}$, $\ell_n \in \mathbb{N}$, and $\phi_n \colon 2^n \to 2^{\ell_n}$, appeal to Lemma 3.2 to obtain a sequence $t_n \in 2^{<\mathbb{N}}$ such that $\mathcal{N}_{\phi_n(s) \frown t_n} \subseteq U_n$ for all $s \in 2^n$. Then there exists $m_n \in \mathbb{N}$ for which $k_{m_n} = f(k_n)$ and $\phi_n(s_n) \frown t_n \sqsubseteq s_{m_n}$, as well as a unique extension $u_n \in 2^{m_n - \ell_n}$ of t_n such that $s_{m_n} = \phi_n(s_n) \frown u_n$. Set $\ell_{n+1} = m_n + 1$, and define $\phi_{n+1} \colon 2^{n+1} \to 2^{\ell_{n+1}}$ by $\phi_{n+1}(t \frown (i)) = \phi_n(t) \frown u_n \frown (i)$ for all $i < 2$ and $t \in 2^n$.

Define $\phi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $\phi(c) = \bigcup_{n\in\mathbb{N}} \phi_n(c \upharpoonright n)$ for all $c \in 2^{\mathbb{N}}$. To see that $\phi(2^{\mathbb{N}}) \subseteq C$, note that if $c \in 2^{\mathbb{N}}$, then $\phi(c) \in \mathcal{N}_{\phi_{n+1}(c\upharpoonright(n+1))} \subseteq \mathcal{N}_{\phi_n(c\upharpoonright n)\frown t_n} \subseteq U_n$ for all $n \in \mathbb{N}$, thus $\phi(c) \in \bigcap_{n\in\mathbb{N}} U_n \subseteq C$. Given $k \in \mathbb{N}$, to see that ϕ is a homomorphism from $\mathbb{G}_{0,k}$ to $\mathbb{G}_{0,f(k)}$, observe that if $c \in 2^{\mathbb{N}}$, $n \in \mathbb{N}$, $k = k_n$, and $d = \bigoplus_{m\in\mathbb{N}} u_{n+1+m} \frown (c(m))$, then $\phi(s_n \frown (i) \frown c) = \phi_{n+1}(s_n \frown (i)) \frown d = \phi_n(s_n) \frown u_n \frown (i) \frown d = s_{m_n} \frown (i) \frown d$ for all $i < 2$, so the fact that $k_{m_n} = f(k_n)$ ensures that $\phi(s_n \frown (0) \frown c)\ \mathbb{G}_{0,f(k_n)}\ \phi(s_n \frown (1) \frown c)$. ⊠

For all $n \in \mathbb{N}$, define $X^{<n} = \bigcup_{m<n} X^m$. For all $s, t \in 2^{<\mathbb{N}}$, define $\mathbb{G}_{s,t} = \{(s \frown (i) \frown t \frown c)_{i<2} \mid c \in 2^{\mathbb{N}}\}$.

Proposition 3.3. *Suppose that $(R_{j,n})_{j\in\mathbb{N},n>0}$ is a sequence of analytic binary relations on $2^{\mathbb{N}}$ such that $\bigcup_{m<n} \mathbb{G}_{0,k_m} \subseteq \bigcup_{j\in\mathbb{N}} R_{j,n}$ for all $n > 0$. Then there are functions $g_n\colon 2^{<n} \to \mathbb{N}$ and a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $(\mathbb{G}_{0,k})_{k\in\mathbb{N}}$ to $(\mathbb{G}_{0,k})_{k\in\mathbb{N}}$ that is also a homomorphism from $(\mathbb{G}_{s_{n-|t|-1},t})_{n>0,t\in 2^{<n}}$ to $(R_{g_n(t),n})_{n>0,t\in 2^{<n}}$.*

Proof. We will recursively construct $m_n \in \mathbb{N}$ and $u_n \in 2^{<\mathbb{N}}$ for all $n \in \mathbb{N}$, from which we define $\phi_{[m,n)}\colon 2^{n-m} \to 2^{<\mathbb{N}}$ by $\phi_{[m,n)}(t) = \bigoplus_{i<n-m} u_{i+m} \frown (t(i))$ for all natural numbers $m \le n$ and sequences $t \in 2^{n-m}$, as well as $g_n\colon 2^{<n} \to \mathbb{N}$ and open sets $U_{j,n} \subseteq 2^{\mathbb{N}}$ for all $j \in \mathbb{N}$ and $n > 0$, satisfying the following conditions:

1. $\forall j \in \mathbb{N} \forall n > 0\ U_{j,n}$ is dense in $\mathcal{N}_{u_n}$.
2. $\forall n > 0 \forall c \in \bigcap_{j\in\mathbb{N}} U_{j,n} \forall t \in 2^{<n}$
   $(\phi_{[0,n)}(s_{n-|t|-1} \frown (i) \frown t) \frown c)_{i<2} \in R_{g_n(t),n}$.
3. $\forall n > 0 \forall t \in 2^{<n}\ \mathcal{N}_{\phi_{[n-|t|,n)}(t)\frown u_n} \subseteq U_{|t|,n-|t|}$.
4. $\forall n \in \mathbb{N}\ (k_{m_n} = k_n$ and $s_{m_n} = \phi_{[0,n)}(s_n) \frown u_n)$.

We begin by setting $m_0 = 0$ and $u_0 = \emptyset$. Suppose now that $n > 0$ and we have already found $(m_k)_{k<n}$ and $(u_k)_{k<n}$, as well as $(g_k)_{0<k<n}$ and $(U_{j,k})_{j\in\mathbb{N},0<k<n}$, satisfying the corresponding fragments of the above conditions. For all $g\colon 2^{<n} \to \mathbb{N}$, let B_g be the set of $c \in 2^{\mathbb{N}}$ such that $(\phi_{[0,n)}(s_{n-|t|-1} \frown (i) \frown t) \frown c)_{i<2} \in R_{g(t),n}$ for all $t \in 2^{<n}$. Note that if $c \in 2^{\mathbb{N}}$, $i < 2$, and $t \in 2^{<n}$, then condition (4) ensures that

$$\begin{aligned}
&\phi_{[0,n)}(s_{n-|t|-1} \frown (i) \frown t) \frown c \\
&\quad = \phi_{[0,n-|t|)}(s_{n-|t|-1} \frown (i)) \frown \phi_{[n-|t|,n)}(t) \frown c \\
&\quad = \phi_{[0,n-|t|-1)}(s_{n-|t|-1}) \frown u_{n-|t|-1} \frown (i) \frown \phi_{[n-|t|,n)}(t) \frown c \\
&\quad = s_{m_{n-|t|-1}} \frown (i) \frown \phi_{[n-|t|,n)}(t) \frown c,
\end{aligned}$$

so $(\phi_{[0,n)}(s_{n-|t|-1} \frown (i) \frown t) \frown c)_{i<2} \in \mathbb{G}_{0,k_{n-|t|-1}} \subseteq \bigcup_{j\in\mathbb{N}} R_{j,n}$, thus there exists $g\colon 2^{<n} \to \mathbb{N}$ for which $c \in B_g$. Fix $g_n\colon 2^{<n} \to \mathbb{N}$ for which

B_{g_n} is non-meager, $u_{0,n} \in 2^{<\mathbb{N}}$ for which B_{g_n} is comeager in $\mathcal{N}_{u_{0,n}}$, and dense open sets $U_{j,n} \subseteq \mathcal{N}_{u_{0,n}}$ such that $\bigcap_{j \in \mathbb{N}} U_{j,n} \subseteq B_{g_n}$. Note that if $t \in 2^{<n}$ and $u \in 2^{<\mathbb{N}}$ extends $u_{0,n}$, then $U_{|t|,n-|t|}$ is dense in $\mathcal{N}_{\phi_{[n-|t|,n)} \frown u}$ (by our choice of $U_{0,n}$ when $t = \emptyset$, and by condition (1) and the fact that $u_{n-|t|} \sqsubseteq \phi_{[n-|t|,n)}(t)$ when $t \neq \emptyset$). Fix an enumeration $(t_{k,n})_{k<2^n-1}$ of $2^{<n}$, and recursively find extensions $u_{k+1,n} \in 2^{<\mathbb{N}}$ of $u_{k,n}$ such that $\mathcal{N}_{\phi_{[n-|t_{k,n}|,n)}(t_{k,n}) \frown u_{k+1,n}} \subseteq U_{|t_{k,n}|,n-|t_{k,n}|}$ for all $k < 2^n - 1$. Fix $m_n \in \mathbb{N}$ for which $k_{m_n} = k_n$ and $\phi_{[0,n)}(s_n) \frown u_{2^n-1,n} \sqsubseteq s_{m_n}$, and let u_n be the unique extension of $u_{2^n-1,n}$ for which $s_{m_n} = \phi_{[0,n)}(s_n) \frown u_n$.

Define $\phi_{[m,\infty)} \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ by $\phi_{[m,\infty)}(c) = \bigcup_{n \in \mathbb{N}} \phi_{[m,m+n)}(c \upharpoonright n)$ for all $c \in 2^{\mathbb{N}}$ and $m \in \mathbb{N}$.

To see that $\phi_{[0,\infty)}$ is a homomorphism from $\mathbb{G}_{0,k}$ to $\mathbb{G}_{0,k}$ for all $k \in \mathbb{N}$, note that if $c \in 2^{\mathbb{N}}$, $n \in \mathbb{N}$, $k = k_n$, and $d = \bigoplus_{m \in \mathbb{N}} u_{n+1+m} \frown (c(m))$, then condition (4) ensures that $\phi_{[0,\infty)}(s_n \frown (i) \frown c) = \phi_{[0,n+1)}(s_n \frown (i)) \frown d = \phi_{[0,n)}(s_n) \frown u_n \frown (i) \frown d = s_{m_n} \frown (i) \frown d$ for all $i < 2$, so $\phi_{[0,\infty)}(s_n \frown (0) \frown c) \; \mathbb{G}_{0,k} \; \phi_{[0,\infty)}(s_n \frown (1) \frown c)$.

Given $n > 0$ and $t \in 2^{<n}$, to see that $\phi_{[0,\infty)}$ is a homomorphism from $\mathbb{G}_{s_{n-|t|-1},t}$ to $R_{g_n(t),n}$, note that if $c \in 2^{\mathbb{N}}$, then condition (3) ensures that $\mathcal{N}_{\phi_{[n,m)}(c \upharpoonright (m-n)) \frown u_m} \subseteq U_{m-n,n}$ for all $m \geq n$, so $\phi_{[n,\infty)}(c) \in \bigcap_{j \in \mathbb{N}} U_{j,n}$. As $\phi_{[0,\infty)}(s_{n-|t|-1} \frown (i) \frown t \frown c) = \phi_{[0,n)}(s_{n-|t|-1} \frown (i) \frown t) \frown \phi_{[n,\infty)}(c)$ for all $i < 2$, it follows that $(\phi_{[0,\infty)}(s_{n-|t|-1} \frown (i) \frown t \frown c))_{i<2} \in R_{g_n(t),n}$, by condition (2). ⊠

For all sets N, let $[N]^{<\aleph_0}$ denote the family of finite subsets of N. For all sequences $c, d \in 2^N$, set $\Delta(c,d) = \{n \in N \mid c(n) \neq d(n)\}$. Define $\delta_i \colon 2^{\mathbb{N} \times \mathbb{N}} \times 2^{\mathbb{N} \times \mathbb{N}} \to \mathbb{N} \cup \{\aleph_0\}$ by $\delta_i(c,d) = |\Delta(c,d) \cap (\{i\} \times \mathbb{N})|$ for all $c, d \in 2^{\mathbb{N} \times \mathbb{N}}$ and $i \in \mathbb{N}$.

A *homomorphism* from a function $f \colon X \times X \to N$ to a function $g \colon Y \times Y \to N$ is a map $\phi \colon X \to Y$ such that $f(w,x) = g(\phi(w), \phi(x))$ for all $w, x \in X$. More generally, a *homomorphism* from a sequence $(f_i \colon X \times X \to N)_{i \in I}$ to a sequence $(g_i \colon Y \times Y \to N)_{i \in I}$ is a map $\phi \colon X \to Y$ that is a homomorphism from f_i to g_i for all $i \in I$.

Proposition 3.4. *Suppose that $C \subseteq 2^{\mathbb{N} \times \mathbb{N}}$ is comeager. Then there is a continuous homomorphism $\phi \colon 2^{\mathbb{N} \times \mathbb{N}} \to C$ from $(\delta_i)_{i \in \mathbb{N}}$ to $(\delta_i)_{i \in \mathbb{N}}$.*

Proof. Fix dense open sets $U_n \subseteq 2^{\mathbb{N} \times \mathbb{N}}$ for which $\bigcap_{n \in \mathbb{N}} U_n \subseteq C$.

Lemma 3.5. *For all $F, G \in [\mathbb{N} \times \mathbb{N}]^{<\aleph_0}$, $\phi \colon 2^F \to 2^G$, and $n \in \mathbb{N}$, there exist $H \in [\sim G]^{<\aleph_0}$ and $t \in 2^H$ such that $\mathcal{N}_{\phi(s) \cup t} \subseteq U_n$ for all $s \in 2^F$.*

Proof. Fix an enumeration $(s_m)_{m<2^{|F|}}$ of 2^F, and recursively find pairwise disjoint sets $H_m \in [\sim G]^{<\aleph_0}$ and $t_m \in 2^{H_m}$ with $\mathcal{N}_{\phi(s_m) \cup \bigcup_{\ell \leq m} t_\ell} \subseteq U_n$ for all $m < 2^{|F|}$. Define $H = \bigcup_{m<2^{|F|}} H_m$ and $t = \bigcup_{m<2^{|F|}} t_m$. ⊠

Fix an injective enumeration $(i_n, j_n)_{n \in \mathbb{N}}$ of $\mathbb{N} \times \mathbb{N}$, and for all $n \in \mathbb{N}$, set $F_n = \{(i_m, j_m) \mid m < n\}$. Set $G_0 = \emptyset$, and define $\phi_0 \colon 2^{F_0} \to 2^{G_0}$ by $\phi_0(\emptyset) = \emptyset$. Given $n \in \mathbb{N}$, a set $G_n \in [\mathbb{N} \times \mathbb{N}]^{<\aleph_0}$, and a function $\phi_n \colon 2^{F_n} \to 2^{G_n}$, appeal to Lemma 3.5 to obtain $H_n \in [\sim G_n]^{<\aleph_0}$ and $t_n \in 2^{H_n}$ such that $\mathcal{N}_{\phi_n(s) \cup t_n} \subseteq U_n$ for all $s \in 2^{F_n}$, fix $k_n \in \mathbb{N}$ for which $(i_n, k_n) \notin G_n \cup H_n$, set $G_{n+1} = G_n \cup H_n \cup \{(i_n, k_n)\}$, and define $\phi_{n+1} \colon 2^{F_{n+1}} \to 2^{G_{n+1}}$ by $\phi_{n+1}(s) \restriction G_n = \phi_n(s \restriction F_n)$, $\phi_{n+1}(s) \restriction H_n = t_n$, and $\phi_{n+1}(s)(i_n, k_n) = s(i_n, j_n)$ for all $s \in 2^{F_{n+1}}$.

Set $G_\infty = \bigcup_{n \in \mathbb{N}} G_n$, and let $\phi \colon 2^{\mathbb{N} \times \mathbb{N}} \to 2^{\mathbb{N} \times \mathbb{N}}$ be the function given by $\mathrm{supp}(\phi(c)) \subseteq G_\infty$ and $\phi(c) \restriction G_\infty = \bigcup_{n \in \mathbb{N}} \phi_n(c \restriction F_n)$ for all $c \in 2^{\mathbb{N} \times \mathbb{N}}$. To see that $\phi(2^{\mathbb{N} \times \mathbb{N}}) \subseteq C$, note that if $c \in 2^{\mathbb{N} \times \mathbb{N}}$, then $\phi(c) \in \mathcal{N}_{\phi_{n+1}(c \restriction F_{n+1})} \subseteq \mathcal{N}_{\phi_n(c \restriction F_n) \cup t_n} \subseteq U_n$ for all $n \in \mathbb{N}$, thus $\phi(c) \in \bigcap_{n \in \mathbb{N}} U_n \subseteq C$. To see that ϕ is a homomorphism from δ_i to δ_i for all $i \in \mathbb{N}$, note that if $m < n$, then $(i_m, k_m) \neq (i_n, k_n)$, since $(i_m, k_m) \in G_{m+1} \subseteq G_n$ but $(i_n, k_n) \notin G_n$, and $\Delta(\phi(c), \phi(d)) = \{(i_n, k_n) \mid n \in \mathbb{N} \text{ and } (i_n, j_n) \in \Delta(c, d)\}$ for all $c, d \in 2^{\mathbb{N} \times \mathbb{N}}$. ⊠

Given $F \subseteq \mathbb{N} \times \mathbb{N}$ and $i \in \mathbb{N}$, set $\Delta_i(c, d) = \Delta(c, d) \cap (i \times \mathbb{N})$ for all $c, d \in 2^F$ and define $\mathbb{D}_{i,F} = \{(c, d) \in 2^{\mathbb{N} \times \mathbb{N}} \times 2^{\mathbb{N} \times \mathbb{N}} \mid \Delta_i(c, d) = F\}$. For all $k \in \mathbb{N}$, let $(=_{2^\mathbb{N}})^k \times \mathbb{E}_0^{<\mathbb{N}} \times (=_{2^\mathbb{N}})^\mathbb{N}$ denote the equivalence relation on $(2^\mathbb{N})^\mathbb{N}$ with respect to which two sequences $c, d \in (2^\mathbb{N})^\mathbb{N}$ are equivalent if and only if there exists $m \geq k$ with the property that $c(\ell) \neq d(\ell) \implies (k \leq \ell < m \text{ and } c(\ell) \mathrel{\mathbb{E}_0} d(\ell))$ for all $\ell \in \mathbb{N}$. We will abuse notation by identifying each binary relation on $(2^\mathbb{N})^\mathbb{N}$ with the corresponding binary relation on $2^{\mathbb{N} \times \mathbb{N}}$.

Proposition 3.6. *Suppose that $D \subseteq 2^{\mathbb{N} \times \mathbb{N}} \times 2^{\mathbb{N} \times \mathbb{N}}$ is closed and nowhere dense in $\mathbb{D}_{i,F}$ and $R \subseteq 2^{\mathbb{N} \times \mathbb{N}} \times 2^{\mathbb{N} \times \mathbb{N}}$ is meager in $\mathbb{D}_{i,F}$, for all $i \in \mathbb{N}$ and $F \in [i \times \mathbb{N}]^{<\aleph_0}$. Then there is a continuous homomorphism $\phi \colon 2^{\mathbb{N} \times \mathbb{N}} \to 2^{\mathbb{N} \times \mathbb{N}}$ from $((=_{2^\mathbb{N}})^k \times \mathbb{E}_0^{<\mathbb{N}} \times (=_{2^\mathbb{N}})^\mathbb{N})_{k \in \mathbb{N}}$ to $((=_{2^\mathbb{N}})^k \times \mathbb{E}_0^{<\mathbb{N}} \times (=_{2^\mathbb{N}})^\mathbb{N})_{k \in \mathbb{N}}$ that is also a homomorphism from $(\neq_{2^{\mathbb{N} \times \mathbb{N}}}, \sim\mathbb{E}_0^\mathbb{N})$ to $(\sim D, \sim R)$.*

Proof. As the function $f \colon 2^{\mathbb{N} \times \mathbb{N}} \times 2^{\mathbb{N} \times \mathbb{N}} \to 2^{\mathbb{N} \times \mathbb{N}} \times 2^{\mathbb{N} \times \mathbb{N}}$, given by $f(c, d) = (d, c)$, is a homeomorphism under which each $\mathbb{D}_{i,F}$ is invariant, it follows that each $f \restriction \mathbb{D}_{i,F}$ is a homeomorphism, so $f(D)$ is closed and nowhere dense in each $\mathbb{D}_{i,F}$, thus by replacing D with $D \cup f(D)$, we can assume that D is symmetric. For all $i \in \mathbb{N}$ and $F \in [i \times \mathbb{N}]^{<\aleph_0}$, fix a decreasing sequence $(U_{i,F,n})_{n \in \mathbb{N}}$ of dense open subsets of $\mathbb{D}_{i,F} \setminus D$ whose intersection is disjoint from R. As each $f(U_{i,F,n})$ is a dense open subset of $\mathbb{D}_{i,F} \setminus D$, by replacing each $U_{i,F,N}$ with $U_{i,F,N} \cap f(U_{i,F,n})$, we can assume that each $U_{i,F,n}$ is symmetric.

Lemma 3.7. *For all $F, G \in [\mathbb{N} \times \mathbb{N}]^{<\aleph_0}$, $\phi \colon 2^F \to 2^G$, and $i, n \in \mathbb{N}$, there exist $H \in [\sim G]^{<\aleph_0}$ and $t_0, t_1 \in 2^H$ with the property that*

$\Delta_i(t_0, t_1) = \emptyset$ *and* $\mathbb{D}_{i,\Delta_i(\phi(s_0),\phi(s_1))} \cap \prod_{k<2} \mathcal{N}_{\phi(s_k)\cup t_k} \subseteq U_{i,\Delta_i(\phi(s_0),\phi(s_1)),n}$ *for all* $s_0, s_1 \in 2^F$.

Proof. Fix an enumeration $(s_{0,m}, s_{1,m})_{m<4^{|F|}}$ of $2^F \times 2^F$, and recursively find pairwise disjoint sets $H_m \in [\sim G]^{<\aleph_0}$ and $t_{0,m}, t_{1,m} \in 2^{H_m}$ such that, for all $m < 4^{|F|}$, the set $\Delta_i(t_{0,m}, t_{1,m})$ is empty and

$$\mathbb{D}_{i,\Delta_i(\phi(s_{0,m}),\phi(s_{1,m}))} \cap \prod_{k<2} \mathcal{N}_{\phi(s_{k,m})\cup\bigcup_{\ell\leq m} t_{k,\ell}} \subseteq U_{i,\Delta_i(\phi(s_{0,m}),\phi(s_{1,m})),n}.$$

Define $H = \bigcup_{m<4^{|F|}} H_m$ and $t_k = \bigcup_{m<4^{|F|}} t_{k,m}$ for all $k < 2$. ⊠

Fix an injective enumeration $(i_n, j_n)_{n\in\mathbb{N}}$ of $\mathbb{N} \times \mathbb{N}$, and for all $n \in \mathbb{N}$, set $F_n = \{(i_m, j_m) \mid m < n\}$. Set $G_0 = \emptyset$, and define $\phi_0 \colon 2^{F_0} \to 2^{G_0}$ by $\phi_0(\emptyset) = \emptyset$. Given $n \in \mathbb{N}$, a set $G_n \in [\mathbb{N} \times \mathbb{N}]^{<\aleph_0}$, and a function $\phi_n \colon 2^{F_n} \to 2^{G_n}$, appeal to Lemma 3.7 to obtain $H_n \in [\sim G_n]^{<\aleph_0}$ and $t_{0,n}, t_{1,n} \in 2^{H_n}$ such that $\Delta_{i_n}(t_{0,n}, t_{1,n}) = \emptyset$ and $\mathbb{D}_{i_n,\Delta_{i_n}(\phi_n(s_0),\phi_n(s_1))} \cap \prod_{k<2} \mathcal{N}_{\phi_n(s_k)\cup t_{k,n}} \subseteq U_{i_n,\Delta_{i_n}(\phi_n(s_0),\phi_n(s_1)),n}$ for all $s_0, s_1 \in 2^{F_n}$, set $G_{n+1} = G_n \cup H_n$, and define $\phi_{n+1} \colon 2^{F_{n+1}} \to 2^{G_{n+1}}$ by $\phi_{n+1}(s) \restriction G_n = \phi_n(s \restriction F_n)$ and $\phi_{n+1}(s) \restriction H_n = t_{s(i_n,j_n),n}$.

Set $G_\infty = \bigcup_{n\in\mathbb{N}} G_n$, and let $\phi \colon 2^{\mathbb{N}\times\mathbb{N}} \to 2^{\mathbb{N}\times\mathbb{N}}$ be the function given by $\operatorname{supp}(\phi(c)) \subseteq G_\infty$ and $\phi(c) \restriction G_\infty = \bigcup_{n\in\mathbb{N}} \phi_n(c \restriction F_n)$ for all $c \in 2^{\mathbb{N}\times\mathbb{N}}$.

To see that ϕ is a homomorphism from $\neq_{2^{\mathbb{N}\times\mathbb{N}}}$ to $\sim D$, note that if $c, d \in 2^{\mathbb{N}\times\mathbb{N}}$ are distinct, then there exists $n \in \mathbb{N}$ with the property that $c(i_n, j_n) \neq d(i_n, j_n)$, so $(\phi(c), \phi(d)) \in U_{i_n,\Delta_{i_n}(\phi_n(c\restriction F_n),\phi_n(d\restriction F_n)),n}$, thus $(\phi(c), \phi(d)) \notin D$.

To see that ϕ is a homomorphism from $(=_{2^{\mathbb{N}}})^k \times \mathbb{E}_0^{<\mathbb{N}} \times (=_{2^{\mathbb{N}}})^{\mathbb{N}}$ to $(=_{2^{\mathbb{N}}})^k \times \mathbb{E}_0^{<\mathbb{N}} \times (=_{2^{\mathbb{N}}})^{\mathbb{N}}$ for all $k \in \mathbb{N}$, note that if $c, d \in 2^{\mathbb{N}\times\mathbb{N}}$ are $((=_{2^{\mathbb{N}}})^k \times \mathbb{E}_0^{<\mathbb{N}} \times (=_{2^{\mathbb{N}}})^{\mathbb{N}})$-related, then $\Delta_k(t_{c(i_n,j_n),n}, t_{d(i_n,j_n),n}) = \emptyset$ for all $n \in \mathbb{N}$, and $c(i_n, j_n) = d(i_n, j_n)$ for all but finitely many $n \in \mathbb{N}$, so $\Delta(\phi(c), \phi(d))$ is a finite subset of $(\mathbb{N} \setminus k) \times \mathbb{N}$.

To see that ϕ is a homomorphism from $\sim\mathbb{E}_0^{\mathbb{N}}$ to $\sim R$, observe that if $c, d \in 2^{\mathbb{N}\times\mathbb{N}}$ are $\mathbb{E}_0^{\mathbb{N}}$-inequivalent, then there is a least $k \in \mathbb{N}$ for which $\delta_k(c, d) = \aleph_0$. Set $F = \Delta_k(c, d)$, fix $n \in \mathbb{N}$ sufficiently large that $F \subseteq F_n$, define $G = \Delta_k(\phi_n(c \restriction F_n), \phi_n(d \restriction F_n))$, and observe that $\Delta_k(\phi(c), \phi(d)) = G$. As there are arbitrarily large $m \geq n$ for which $i_m = k$ and $c(i_m, j_m) \neq d(i_m, j_m)$, and therefore $(\phi(c), \phi(d)) \in U_{k,G,m}$, it follows that $(\phi(c), \phi(d)) \notin R$. ⊠

A subset of a topological space is F_σ if it is a countable union of closed sets, and G_δ if it is a countable intersection of open sets. For all sets N and sequences $c \in 2^N$, let $\overline{c}$ denote the element of 2^N given by $\overline{c}(n) = 1 - c(n)$ for all $n \in N$. As the Lusin-Novikov uniformization theorem and standard change of topology results (see, for example, [Kec95, §13]) ensure that every countable Borel equivalence relation on a Polish space is Borel isomorphic to an F_σ Borel equivalence relation on a Polish space, the following well-known fact ensures that $\mathbb{E}_0^{\mathbb{N}}$ is not essentially countable:

Proposition 3.8. *Suppose that X is a second countable space and E is an F_σ equivalence relation on X. Then there is no Baire measurable reduction $\phi\colon 2^{\mathbb{N}\times\mathbb{N}} \to X$ of $\mathbb{E}_0^{\mathbb{N}}$ to E.*

Proof. Suppose that $\phi\colon 2^{\mathbb{N}\times\mathbb{N}} \to X$ is a Baire measurable homomorphism from $\sim\mathbb{E}_0^{\mathbb{N}}$ to $\sim E$. Then there exist a dense G_δ set $C \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ for which $\phi \restriction C$ is continuous (see, for example, [Kec95, Proposition 8.38]), dense open sets $U_n \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ such that $C = \bigcap_{n\in\mathbb{N}} U_n$, and open sets $V_n \subseteq X \times X$ with the property that $\sim E = \bigcap_{n\in\mathbb{N}} V_n$.

Fix an enumeration $(i_n, j_n)_{n\in\mathbb{N}}$ of $\mathbb{N}\times\mathbb{N}$. Set $F_0 = \emptyset$ and $s_0 = t_0 = \emptyset$. Given $n \in \mathbb{N}$, $F_n \in [\mathbb{N}\times\mathbb{N}]^{<\aleph_0}$, and $s_n, t_n \in 2^{F_n}$, fix $G_n \in [\sim F_n]^{<\aleph_0}$ and $u_n \in 2^{G_n}$ such that $\mathcal{N}_{s_n\cup u_n} \subseteq U_n$, as well as $H_n \in [\sim(F_n \cup G_n)]^{<\aleph_0}$ and $v_n \in 2^{H_n}$ such that $\mathcal{N}_{t_n\cup u_n\cup v_n} \subseteq U_n$. Set $F'_n = F_n \cup G_n \cup H_n$, $s'_n = s_n \cup u_n \cup v_n$, and $t'_n = t_n \cup u_n \cup v_n$, and define $\psi_n\colon \mathcal{N}_{s'_n} \to \mathcal{N}_{t'_n}$ by $\psi_n(c \cup d \cup s'_n) = c \cup \overline{d} \cup t'_n$ for all $c \in 2^{(n\times\mathbb{N})\setminus F'_n}$ and $d \in 2^{((\sim n)\times\mathbb{N})\setminus F'_n}$. Fix $c_n \in C \cap \psi_n^{-1}(C)$ and set $d_n = \psi_n(c_n)$. As these points are $\mathbb{E}_0^{\mathbb{N}}$-inequivalent, it follows that the points $x_n = \phi(c_n)$ and $y_n = \phi(d_n)$ are E-inequivalent, and therefore V_n-related. Fix $F_{n+1} \in [\mathbb{N}\times\mathbb{N}]^{<\aleph_0}$ such that $F'_n \cup \{(i_n, j_n)\} \subseteq F_{n+1}$ and $\phi(C \cap \mathcal{N}_{c_n \restriction F_{n+1}}) \times \phi(C \cap \mathcal{N}_{d_n \restriction F_{n+1}}) \subseteq V_n$, and define $s_{n+1} = c_n \restriction F_{n+1}$ and $t_{n+1} = d_n \restriction F_{n+1}$.

The fact that $\mathcal{N}_{s_{n+1}}, \mathcal{N}_{t_{n+1}} \subseteq U_n$ for all $n \in \mathbb{N}$ ensures that the sequences $c = \bigcup_{n\in\mathbb{N}} s_n$ and $d = \bigcup_{n\in\mathbb{N}} t_n$ are in C, in which case $(\phi(c), \phi(d)) \in \phi(C \cap \mathcal{N}_{s_{n+1}}) \times \phi(C \cap \mathcal{N}_{t_{n+1}}) \subseteq V_n$ for all $n \in \mathbb{N}$, thus $\phi(c)$ and $\phi(d)$ are E-inequivalent. As the fact that $\Delta_n(c, d) = \Delta_n(s_n, t_n)$ for all $n \in \mathbb{N}$ ensures that $c \mathrel{\mathbb{E}_0^{\mathbb{N}}} d$, it follows that ϕ is not a homomorphism from $\mathbb{E}_0^{\mathbb{N}}$ to E. ⊠

4. A strengthening of the $\mathbb{E}_0^{\mathbb{N}}$ dichotomy

We begin this section with the following:

Proof of Theorem 2. To see that conditions (1) and (2) are mutually exclusive when $(\Delta_k)_{k\in\mathbb{N}}$ is a decreasing sequence of conjugation-invariant sets, observe that if $\Gamma \curvearrowright X$ is σ-lacunary and $(\Delta'_k)_{k\in\mathbb{N}}$ is a subsequence of $(\Delta_k)_{k\in\mathbb{N}}$, then $(\Delta'_k)_{k\in\mathbb{N}}$ is also a neighborhood basis of 1_Γ, so Proposition 2.2 gives rise to Borel sets $B_i \subseteq X$ such that $X = \bigcup_{i\in\mathbb{N}} B_i$ and $\forall i, j \in \mathbb{N}\ \chi_B((R^X_{\Delta'_i} \setminus R^X_{\Delta'_j}) \restriction B_i) \leq \aleph_0$, in which case Theorem 1 rules out the existence of a continuous homomorphism from $(\mathbb{G}_{0,k})_{k\in\mathbb{N}}$ to $(R^X_{\Delta'_k} \setminus R^X_{\Delta'_{k+1}})_{k\in\mathbb{N}}$.

To see that at least one of the conditions holds, note first that condition (2) is equivalent to the apparently weaker statement in which ϕ is merely Borel, since we can always pass to a dense G_δ set $C \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ on which ϕ is continuous (see, for example, [Kec95, Theorem 8.38]), and then compose $\phi \upharpoonright C$ with the function obtained by applying Proposition 3.1 to C and the identity function. By [BK96, Theorem 5.2.1], we can therefore assume that $\Gamma \curvearrowright X$ is continuous. We can also assume that Γ is not discrete, since otherwise $\Gamma \curvearrowright X$ is trivially σ-lacunary. So, by passing to a subsequence, we can assume that $(\Delta_k)_{k\in\mathbb{N}}$ is decreasing. By Theorem 1 and Proposition 2.5, it is therefore sufficient to show that if there exist $f\colon \mathbb{N} \to \mathbb{N}$ and a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $(\mathbb{G}_{0,k})_{k\in\mathbb{N}}$ to $(R^X_{\Delta_k} \setminus R^X_{\Delta_{f(k)}})_{k\in\mathbb{N}}$, then condition (2) holds. Towards this end, note that $f(k) > k$ for all $k \in \mathbb{N}$, and let $(\Delta'_k)_{k\in\mathbb{N}}$ be the subsequence of $(\Delta_k)_{k\in\mathbb{N}}$ given by $\Delta'_k = \Delta_{f^k(0)}$ for all $k \in \mathbb{N}$. By one more application of Proposition 3.1, there is a continuous homomorphism $\psi\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $(\mathbb{G}_{0,k})_{k\in\mathbb{N}}$ to $(\mathbb{G}_{0,f^k(0)})_{k\in\mathbb{N}}$, in which case $\phi \circ \psi$ is a continuous homomorphism from $(\mathbb{G}_{0,k})_{k\in\mathbb{N}}$ to $(R^X_{\Delta'_k} \setminus R^X_{\Delta'_{k+1}})_{k\in\mathbb{N}}$. ⊠

As a corollary, we obtain an approximation to Theorem 3 that goes through for all tsi Polish groups:

Theorem 4.1. *Suppose that Γ is a tsi Polish group, $(\Gamma_k)_{k\in\mathbb{N}}$ is a sequence of open subgroups of Γ, X is a Polish space, $\Gamma \curvearrowright X$ is Borel, and R^X_Δ is Borel for all open sets $\Delta \subseteq \Gamma$. Then at least one of the following holds:*

(1) *The action $\Gamma \curvearrowright X$ is σ-lacunary.*
(2) *There is a continuous injective homomorphism $\phi\colon 2^{\mathbb{N}\times\mathbb{N}} \to X$ from $((=_{2^{\mathbb{N}}})^k \times \mathbb{E}_0^{<\mathbb{N}} \times (=_{2^{\mathbb{N}}})^{\mathbb{N}})_{k\in\mathbb{N}}$ to $(E^X_{\Gamma_k})_{k\in\mathbb{N}}$ that is also a homomorphism from $\sim\mathbb{E}_0^{\mathbb{N}}$ to $\sim E^X_\Gamma$.*

Proof. Fix a neighborhood basis $(\Delta_k)_{k\in\mathbb{N}}$ of 1_G and a conjugation-invariant open neighborhood $\Delta''_0 \subseteq \Gamma_0 \cap \Delta_0$ of 1_Γ, and set $\Delta'_0 = \Delta''_0 \cap (\Delta''_0)^{-1}$. Given $k \in \mathbb{N}$ and an open neighborhood $\Delta'_k \subseteq \Gamma_k \cap \Delta_k$ of 1_Γ, fix an open neighborhood $\Delta'''_{k+1} \subseteq \Gamma_{k+1} \cap \Delta_{k+1}$ of 1_Γ such that $(\Delta'''_{k+1})^2 \subseteq \Delta'_k$, as well as a conjugation-invariant open neighborhood $\Delta''_{k+1} \subseteq \Delta'''_{k+1}$ of 1_Γ, and set $\Delta'_{k+1} = \Delta''_{k+1} \cap (\Delta''_{k+1})^{-1}$. By replacing each Δ_k with Δ'_k, we can assume that $\Delta_k \subseteq \Gamma_k$ is a conjugation-invariant symmetric open neighborhood of 1_Γ such that $\Delta^2_{k+1} \subseteq \Delta_k$, for all $k \in \mathbb{N}$.

Theorem 2 ensures that, after replacing $(\Delta_k)_{k\in\mathbb{N}}$ with a subsequence if necessary, it is sufficient to show that if there is a continuous homomorphism $\phi\colon 2^{\mathbb{N}} \to X$ from $(\mathbb{G}_{0,k})_{k\in\mathbb{N}}$ to $(R^X_{\Delta_k} \setminus R^X_{\Delta_{k+1}})_{k\in\mathbb{N}}$, then condition (2) holds.

Fix an enumeration $(\delta_j)_{j\in\mathbb{N}}$ of a countable dense subset of Γ, and for all $j \in \mathbb{N}$ and $n > 0$, define $R'_{j,n} = (\phi\times\phi)^{-1}(R^X_{\delta_j\Delta_n})$. As Proposition 2.1 ensures that $\Gamma = \bigcup_{j\in\mathbb{N}} \delta_j\Delta_n$ for all $n \in \mathbb{N}$, it follows that $\bigcup_{k\in\mathbb{N}} \mathbb{G}_{0,k} \subseteq \bigcup_{j\in\mathbb{N}} R'_{j,n}$ for all $n \in \mathbb{N}$, so Proposition 3.3 yields functions $g_n \colon 2^{<n} \to \mathbb{N}$ and a continuous homomorphism $\psi' \colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ from $(\mathbb{G}_{0,k})_{k\in\mathbb{N}}$ to $(\mathbb{G}_{0,k})_{k\in\mathbb{N}}$ that is also a homomorphism from $(\mathbb{G}_{s_{n-|t|-1},t})_{n>0,t\in 2^{<n}}$ to $(R'_{g_n(t),n})_{n>0,t\in 2^{<n}}$. By replacing ϕ with $\phi\circ\psi'$ and defining $\gamma_n(t) = \delta_{g_n(t)}$ for all $n > 0$ and $t \in 2^{<n}$, we can assume that ϕ is also a homomorphism from $(\mathbb{G}_{s_{n-|t|-1},t})_{n>0,t\in 2^{<n}}$ to $(R^X_{\gamma_n(t)\Delta_n})_{n>0,t\in 2^{<n}}$. For each set $\Delta \subseteq \Gamma$, we use $\langle\Delta\rangle$ to denote the group generated by Δ.

Lemma 4.2. *The function ϕ is a homomorphism from $(\mathbb{G}_s)_{s\in 2^{<\mathbb{N}}}$ to $(E^X_{\langle\Delta_{k_{|s|}}\rangle})_{s\in 2^{<\mathbb{N}}}$.*

Proof. A *graph* on a set V is a symmetric digraph G on V. A *G-path* between points u and v is a sequence $(w_j)_{j\leq\ell}$, where $\ell \in \mathbb{N}$, such that $u = w_0$, $w_j \; G \; w_{j+1}$ for all $j < \ell$, and $w_\ell = v$. A graph G on a set V is *connected* if there is a G-path between any two points of V. For all $n \in \mathbb{N}$, let T_n be the graph on 2^n consisting of all pairs of the form $(s_{n-|t|-1} \frown (i) \frown t, s_{n-|t|-1} \frown (1-i) \frown t)$, where $i < 2$ and $t \in 2^{<n}$.

Sublemma 4.3. *Suppose that $n \in \mathbb{N}$. Then T_n is connected.*

Proof. As the case $n = 0$ is trivial, it is sufficient to show that if $n \in \mathbb{N}$ and T_n is connected, then so too is T_{n+1}. Towards this end, suppose that $u_0, u_1 \in 2^n$, and note that if $i < 2$ and $(t_j)_{j\leq\ell}$ is a T_n-path from u_0 to u_1, then $(t_j \frown (i))_{j\leq\ell}$ is a T_{n+1}-path from $u_0 \frown (i)$ to $u_1 \frown (i)$. Similarly, if $(t_{0,j})_{j\leq\ell_0}$ and $(t_{1,j})_{j\leq\ell_1}$ are T_n-paths from u_0 to s_n and from s_n to u_1, respectively, then $(t_{0,j} \frown (0))_{j\leq\ell_0} \frown (t_{1,j} \frown (1))_{j\leq\ell_1}$ is a T_{n+1}-path from $u_0 \frown (0)$ to $u_1 \frown (1)$. ⊠

Given $n \in \mathbb{N}$ and $s \in 2^n$, fix a T_n-path $(t_j)_{j\leq\ell}$ from s to s_n, and for all $j < \ell$, fix $i_j < 2$ and $u_j \in 2^{<n}$ with the property that $t_j = s_{n-|u_j|-1} \frown (i_j) \frown u_j$ and $t_{j+1} = s_{n-|u_j|-1} \frown (1-i_j) \frown u_j$. Note that if $c \in 2^{\mathbb{N}}$, $i < 2$, and $j < \ell$, then $t_j \frown (i) \frown c$ and $t_{j+1} \frown (i) \frown c$ are $\mathbb{G}_{s_{n-|u_j|-1},u_j}$-related, so $\phi(t_j \frown (i) \frown c)$ and $\phi(t_{j+1} \frown (i) \frown c)$ are $R^X_{\gamma_n(u_j)\Delta_n}$-related, and since $k_n \leq n$, thus $\Delta_n \leq \Delta_{k_n}$, there is an element of $(\gamma_n(u_{\ell-1})\Delta_{k_n} \cdots \gamma_n(u_0)\Delta_{k_n})^{-1}\Delta_{k_n}(\gamma_n(u_{\ell-1})\Delta_{k_n} \cdots \gamma_n(u_0)\Delta_{k_n})$ sending $\phi(s \frown (0) \frown c)$ to $\phi(s \frown (1) \frown c)$. As the conjugation invariance and symmetry of Δ_{k_n} ensure that this product is $\Delta^{2\ell+1}_{k_n}$, it follows that $\phi(s \frown (0) \frown c) \; E^X_{\langle\Delta_{k_n}\rangle} \; \phi(s \frown (1) \frown c)$. ⊠

Set $\ell_n = |\{m < n \mid k_m = k_n\}|$ for all $n \in \mathbb{N}$. Define $\psi \colon 2^{\mathbb{N}\times\mathbb{N}} \to 2^{\mathbb{N}}$ by $\psi(c)(n) = c(k_n, \ell_n)$ for all $c \in 2^{\mathbb{N}\times\mathbb{N}}$ and $n \in \mathbb{N}$.

Lemma 4.4. *The function $\phi \circ \psi$ is a homomorphism from $((=_{2^{\mathbb{N}}})^k \times \mathbb{E}_0^{<\mathbb{N}} \times (=_{2^{\mathbb{N}}})^{\mathbb{N}})_{k \in \mathbb{N}}$ to $(E^X_{\langle \Delta_k \rangle})_{k \in \mathbb{N}}$.*

Proof. By the obvious inductive argument, it is sufficient to establish that if sequences $c, d \in 2^{\mathbb{N} \times \mathbb{N}}$ differ at a single coordinate $(i, j) \in \mathbb{N} \times \mathbb{N}$, then $(\phi \circ \psi)(c) \ E^X_{\langle \Delta_i \rangle} \ (\phi \circ \psi)(d)$. Towards this end, fix $n \in \mathbb{N}$ for which $(i, j) = (k_n, \ell_n)$, set $s = \psi(c) \upharpoonright n = \psi(d) \upharpoonright n$, and observe that $\psi(c) \ \mathbb{G}_s \ \psi(d)$ or $\psi(d) \ \mathbb{G}_s \ \psi(c)$, in which case Lemma 4.2 ensures that $(\phi \circ \psi)(c) \ E^X_{\langle \Delta_{k_n} \rangle} \ (\phi \circ \psi)(d)$, thus $(\phi \circ \psi)(c) \ E^X_{\langle \Delta_i \rangle} \ (\phi \circ \psi)(d)$. ⊠

Set $D = ((\phi \circ \psi) \times (\phi \circ \psi))^{-1}(=_X)$ and $E = ((\phi \circ \psi) \times (\phi \circ \psi))^{-1}(E^X_\Gamma)$.

Lemma 4.5. *Suppose that $i \in \mathbb{N}$ and $F \in [i \times \mathbb{N}]^{<\aleph_0}$. Then E is meager in $\mathbb{D}_{i,F}$.*

Proof. Suppose, towards a contradiction, that E is not meager in $\mathbb{D}_{i,F}$. Then another application of Proposition 2.1 yields $j \in \mathbb{N}$ for which the relation $R_{i+2,j} = ((\phi \circ \psi) \times (\phi \circ \psi))^{-1}(R^X_{\delta_j \Delta_{i+2}})$ is not meager in $\mathbb{D}_{i,F}$, so there exist $G \in [(i \times \mathbb{N}) \setminus F]^{<\aleph_0}$ and $H, H' \in [(\sim i) \times \mathbb{N}]^{<\aleph_0}$ for which there are sequences $r \in 2^F$, $s \in 2^G$, $t \in 2^H$, and $t' \in 2^{H'}$ with the property that $R_{i+2,j}$ is comeager in $\mathbb{D}_{i,F} \cap (\mathcal{N}_{r \cup s \cup t} \times \mathcal{N}_{\overline{r} \cup s \cup t'})$, in which case the set S of $(c, (d, d')) \in 2^{(i \times \mathbb{N}) \setminus (F \cup G)} \times (2^{((\sim i) \times \mathbb{N}) \setminus H} \times 2^{((\sim i) \times \mathbb{N}) \setminus H'})$ for which $((c \cup r \cup s) \cup (d \cup t)) \ R_{i+2,j} \ ((c \cup \overline{r} \cup s) \cup (d' \cup t'))$ is comeager.

Let C be the set of $c \in 2^{(i \times \mathbb{N}) \setminus (F \cup G)}$ for which S_c is comeager, and let D be the set of $(c, d) \in 2^{(i \times \mathbb{N}) \setminus (F \cup G)} \times 2^{((\sim i) \times \mathbb{N}) \setminus H}$ for which $(S_c)_d$ is comeager. The Kuratowski-Ulam theorem ensures that C is comeager, as is D_c for all $c \in C$.

Set $R_{i+1} = ((\phi \circ \psi) \times (\phi \circ \psi))^{-1}(R^X_{\Delta_{i+1}})$, and let T be the set of $(c, (d, e)) \in 2^{(i \times \mathbb{N}) \setminus (F \cup G)} \times (2^{((\sim i) \times \mathbb{N}) \setminus H} \times 2^{((\sim i) \times \mathbb{N}) \setminus H})$ with the property that $((c \cup r \cup s) \cup (d \cup t)) \ R_{i+1} \ ((c \cup r \cup s) \cup (e \cup t))$.

Sublemma 4.6. *Suppose that $c \in C$. Then $D_c \times D_c \subseteq T_c$.*

Proof. Suppose that $d, e \in D_c$. Then there exists $d' \in (S_c)_d \cap (S_c)_e$, so $(\phi \circ \psi)((c \cup \overline{r} \cup s) \cup (d' \cup t')) \in \delta_j \Delta_{i+2}(\phi \circ \psi)((c \cup r \cup s) \cup (f \cup t))$ for all $f \in \{d, e\}$, in which case $(\phi \circ \psi)((c \cup r \cup s) \cup (e \cup t))$ is in $(\delta_j \Delta_{i+2})^{-1} \delta_j \Delta_{i+2}(\phi \circ \psi)((c \cup r \cup s) \cup (d \cup t))$, which is itself contained in $\Delta_{i+1}(\phi \circ \psi)((c \cup r \cup s) \cup (d \cup t))$, thus $d \ T_c \ e$. ⊠

Set $M = \{m \in \mathbb{N} \mid (k_m, \ell_m) \in F \cup G \cup H\}$, and define $u \in 2^M$ by $u(m) = (r \cup s \cup t)(k_m, \ell_m)$ for all $m \in M$. Then there exists $n \in \mathbb{N}$ for which $k_n = i$ and $u \sqsubseteq s_n$. Define $N = \{(k_j, \ell_j) \mid j < n\}$ and $u_n \in 2^N$ by $u_n(k_j, \ell_j) = s_n(j)$ for all $j < n$, and fix $c \in C$ for which $u_n \upharpoonright ((i \times \mathbb{N}) \cap N) \sqsubseteq c \cup r \cup s$. Let $\phi_n \colon 2^{((\sim i) \times \mathbb{N}) \setminus H} \to 2^{((\sim i) \times \mathbb{N}) \setminus H}$ be the homeomorphism flipping coordinate (k_n, ℓ_n), and fix $d \in D_c \cap \phi_n^{-1}(D_c)$

for which $u_n \upharpoonright (((\sim i) \times \mathbb{N}) \cap N) \sqsubseteq d \cup t$. Then Sublemma 4.6 ensures that $((c \cup r \cup s) \cup (d \cup t)) \; R_{i+1} \; ((c \cup r \cup s) \cup (\phi_n(d) \cup t))$, contradicting the fact that ϕ is a homomorphism from $\mathbb{G}_{0,i}$ to $\sim R^X_{\Delta_{i+1}}$. ⊠

By composing $\phi \circ \psi$ with the function obtained from applying Proposition 3.6 to D and E, we obtain the desired homomorphism. ⊠

When Γ is non-archimedean, we obtain the following:

Proof of Theorem 3. To see that conditions (1) and (2) are mutually exclusive, note that if $\Gamma \curvearrowright X$ is σ-lacunary, then the Lusin-Novikov uniformization theorem and Proposition 2.7 ensure that E^X_Γ is essentially countable, so Proposition 3.8 and the remarks preceding it imply that there is no continuous embedding of $\mathbb{E}_0^{\mathbb{N}}$ into E^X_Γ.

It remains to show that at least one of the two conditions holds. By [BK96, Theorem 7.1.2], the orbit equivalence relation induced by every open subgroup of Γ is Borel. The fact that Γ is non-archimedean therefore implies that the orbit relation induced by every open subset of Γ is Borel. Note that condition (2) is equivalent to the apparently weaker statement in which ϕ is merely Borel, since we can always pass to a dense G_δ set $C \subseteq 2^{\mathbb{N} \times \mathbb{N}}$ on which ϕ is continuous, and then compose $\phi \upharpoonright C$ with the function given by Proposition 3.4. By [BK96, Theorem 5.2.1], we can therefore assume that $\Gamma \curvearrowright X$ is continuous. Fix a decreasing neighborhood basis $(\Delta_k)_{k \in \mathbb{N}}$ of 1_Γ of open subgroups of Γ. By replacing each Δ_k with $\Gamma_k \cap \Delta_k$, we can assume that $\Delta_k \subseteq \Gamma_k$ for all $k \in \mathbb{N}$.

By Theorem 4.1, it is sufficient to show that if $\phi \colon 2^{\mathbb{N} \times \mathbb{N}} \to X$ is a continuous homomorphism from $((=_{2^{\mathbb{N}}})^k \times \mathbb{E}_0^{<\mathbb{N}} \times (=_{2^{\mathbb{N}}})^{\mathbb{N}})_{k \in \mathbb{N}}$ to $(E^X_{\Delta_k})_{k \in \mathbb{N}}$, then it is a homomorphism from $((=_{2^{\mathbb{N}}})^k \times \mathbb{E}_0^{\mathbb{N}})_{k \in \mathbb{N}}$ to $(E^X_{\Delta_k})_{k \in \mathbb{N}}$. Towards this end, suppose that $k \in \mathbb{N}$ and $c, d \in 2^{\mathbb{N} \times \mathbb{N}}$ are $((=_{2^{\mathbb{N}}})^k \times \mathbb{E}_0^{\mathbb{N}})$-equivalent, and for all $n \geq k$, let d_n be the element of $2^{\mathbb{N} \times \mathbb{N}}$ that agrees with d on $n \times \mathbb{N}$, and with c off of it. Then $d_n \to d$, so $\phi(d_n) \to \phi(d)$. For all $n \geq k$, fix $\delta_n \in \Delta_n$ such that $\delta_n \cdot \phi(d_n) = \phi(d_{n+1})$. Then $\delta_n \cdots \delta_k \cdot \phi(c) = \phi(d_{n+1})$ for all $n \geq k$, so $\delta_n \cdots \delta_k \cdot \phi(c) \to \phi(d)$. As $\delta_n \cdots \delta_m \in \Delta_m$ for all natural numbers $n \geq m \geq k$, it follows that $(\delta_n \cdots \delta_k)_{n \geq k}$ is Cauchy with respect to every compatible complete right-invariant metric on Γ, and therefore converges to some $\delta \in \Delta_k$ (since open subgroups of topological groups are necessarily closed). Then $\delta_n \cdots \delta_k \cdot \phi(c) \to \delta \cdot \phi(c)$, so $\delta \cdot \phi(c) = \phi(d)$, thus $\phi(c) \; E^X_{\Delta_k} \; \phi(d)$. ⊠

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